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The Nature of Mathematics

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The Nature of Mathematics

A Critical Survey

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PREFACE

THE philosophy of mathematics has suffered from a superfluity of technicalities. This is a pity because it increases the difficulties of acquaintance with that delightful subject. But a more serious consequence is the lack of co-operation and mutual criticism between different groups of experts in this field. In England, for example, the fame of Russell and Whitehead's justly celebrated *Principia Mathematica*, is accompanied by almost complete neglect and ignorance of the equally interesting work of the Formalists and Intuitionists on the Continent. There is much to be said in extenuation for this state of affairs, for the relevant papers are scattered in foreign periodicals, untranslated, often difficult to obtain, and are unintelligible without an extensive acquaintance with the terminology and context of their authors' opinions. To fill this gap in the literature of the nature of mathematics would be a work of many years, and the pages which follow are intended to be no more than an introduction to the whole subject.

I have had two aims in mind: to present a considered critical exposition of *Principia Mathematica* and to give supplementary accounts of the formalist and intuitionist doctrines in sufficient detail to lighten the paths of all who may be provoked to read the original papers. Various innovations have been introduced and, though I have not avoided technicalities where they were necessary, all technical terms and symbols have been as far as possible defined. So I hope this book may be of use not only to specialists in mathematical logic but to philosophers and others who begin to read it with less knowledge of the complexities of symbolism. In order to assist readers who may wish to omit sections chiefly concerned with technicalities or familiar definitions, I have adopted the device of adding to many sections a summary or comment, printed in small type immediately after the corresponding subheadings; and I would encourage readers new to the subject to read the introduction and these scattered comments before reading the remainder of the text.

I wish to express my thanks to Professor Bernays for much helpful information concerning the formalists, to Dr. Chwistek for copies of his papers, to the Aristotelian Society for permission to incorporate part of a paper read in 1933, to Professor L. S. Stebbing, Dr. J. H. Woodger, and Miss M. MacDonald for reading the following pages in proof and for much encouragement, and to S. Black, J. M. Burnett, and L. E. R. Mowat for assistance with the transcription of manuscript.

M. B.

May, 1933.

THE NATURE OF MATHEMATICS

A CRITICAL SURVEY

INTRODUCTION

The task of philosophy, *qua* critic, is to exhibit the structure of the sciences by discriminating between hypotheses and principles, etc.

THE successes of the scientific method have led philosophers to dream of a scientific philosophy which, by borrowing the technique of the established sciences, might hope to reach something of their certainty and cumulative success. Philosophy, however, in its function of critic—and it is with that aspect of philosophy that we shall be here concerned—cannot desire to compete with the sciences. The discovery of empirical generalization is the work of the experimental sciences, the formulation of self-evident laws belongs to mathematics, and both are outside the scope of critical philosophy. Its object is to clarify by criticizing knowledge already organized into systems; and of these it prefers the older, more developed, sciences, which combine extreme complexity of theory with consistency in practice. For these qualities are associated with a high degree of utility in practical applications and induce in the creators and admirers of the science a state of self-consciousness inviting the apologetic services of philosophy. In each of these respects the science of mathematics is a most admirable field for the exercise of applied philosophy.

The implied assertion that the established sciences are highly consistent needs to be qualified by the explicit recognition that no science which is still in the process of developing is more than partially self-consistent. For scientific research is characterized by the choice between mutually inconsistent theories, lack of relevant data leading to the postulation of provisional hypotheses which subsequently require to be limited in their application or even totally abandoned.

Postulates need to be distinguished as hypotheses and principles; for, of those postulates which are not ultimately rejected but are incorporated into the main body of the science as knowledge accumulates, some become theorems or laws while others, through their success in stimulating fruitful research, gradually acquire the character of general principles, which embody concepts fundamental to the science. Hypotheses, that is postulates which may become laws, can be disproved casually enough, but principles, since they control the manner in which problems are formulated or difficulties resolved, are formally not susceptible to disproof, and their rejection requires a violent revolution in the methods of the science.

Vagueness of the concepts which occur in the normative principles makes their exact formulation an ideal which is approached by gradual approximation; clear understanding of the concepts used occurs late in the history of a science.

Postulates and concepts are created not by the common agreement of scientists but by scattered individuals or small groups. At the moment of conception concepts are formless, implications of theories are only partially understood; later, theories produced by specialists in one department of the science are found to conflict with the postulates of other departments, in themselves equally plausible or as firmly established. The necessity of resolving such discords reacts upon the concepts of the science, leads to more exact formulation of the postulates and clearer understanding of the concepts involved. Even at moments of apparently extreme stability, the equilibrium of scientific opinion is the immobility of a body under the action of mutually opposing forces.

This state of affairs is a commonplace in the experience of any scientific researcher, yet it is more than that private conflict of ideas in the inventor's mind which is part of the process of invention. The contradictions inhere in the very principles of the science, produced by the inevitable vagueness of the concepts it employs. However much reflection and experiment by the inventors of theories may mitigate the opposition of mutually contradictory opinions by modification and elimination of obscurity, contradictions remain even in scientific theories which find widespread acceptance. In the theories of all branches of science where progress is still being made, in biology, physics, chemistry, mathematics, there are striking paradoxes and contradictions to be found, and those sciences alone are completely consistent which, like anatomy, have degenerated into catalogues. It is important to recognize and distinguish contradictions produced by imprecise formulation of concepts; they are often a sign of vitality and indicate that the scientist's capacity for recognizing relevance and unity in a confusing multiplicity of heterogeneous phenomena is ahead of the careful expression of its discoveries.

Nowhere have such contradictions been more frequent than in mathematics, nor has progress in any science been more steady. Gauss and Fermat, among scores of other famous names, are sufficient illustrations of famous mathematicians who were able to obtain, by apparently fallacious reasoning, valid results of the highest importance in subsequent mathematical researches.

The title of "The Foundations of Mathematics" which the philosophical analysis of mathematics has often received is therefore a misleading one if, taken in conjunction with these contradictions, it suggests that the traditional certainty of mathematics is in question. It is a fallacy to which the philosopher is particularly liable to imagine that the mathematical edifice is in danger through weak foundations, or that philosophy must be invited like a newer Atlas to carry the burden of the disaster on its shoulders.

The progressive elimination of contradictions in mathematics is the work of mathematical insight, a continuous process which can be clearly traced in successive mathematical researches. Philosophical analysis has the equally valuable aim of exhibiting the structure of mathematics; first, the internal structure, by showing the interdependence of theorems, axioms, and definitions, distinguishing between hypotheses and principles, etc.; secondly, the external structure, the relation of mathematical knowledge to non-mathematical.

Exhibition of internal structure has technical importance for mathematics by leading to the rejection of unnecessary postulates and again to the recognition of unexpected analogies between the anatomies of different mathematical disciplines. Such morphological investigations require mathematical technique, and particularly the extensive use of symbols. For mathematics is the study of all structures whose form can be expressed in symbols, it is the grammar of all symbolic systems and, as such, its methods are peculiarly appropriate to the investigation of its own internal structure. But the structure of mathematics, though implicit in its theorems, is not clearly shown and tends to be confused even by those who are most familiar with it. It is the philosopher's task to exhibit the inherent structure and to invent a suitable symbolism for its expression. Elimination of unnecessary postulates and the explicit exhibition of the structure of mathematics prevents confusion of purpose within the science and adds to the æsthetic satisfaction of contemplating it.

The technique required for this type of analysis does not, in the present writer's opinion, require the acceptance of any metaphysical dogmas; in its systematic aspect it can be correctly regarded as a branch of applied mathematics if that science is not restricted to physical applications but is allowed to include any subject-matter amenable to mathematical investigations; in its philosophic aspect it is a branch of applied logic.[1] The details of such a technique must, however, be reserved for future exposition. The purpose of this essay is only to report and criticize attempts that have already been made to analyze mathematics.

Philosophical analysis must take into account lack of structure for, in so far as a science contains inconsistencies, it cannot be considered as a system, it is to that extent in process of acquiring a form and not in possession of one. Philosophers, however, under scholastic influences, have too often overlooked this fact and have been suspected in consequence by the practising scientist. For, when faced with the difficulty of clarifying existing knowledge, the temptation is great to find compensation in admiring the complex structure which represents partial success and to supplement it by unwarranted extrapolation. In the case of one's own philosophic system familiarity or the inertia of habitual thought processes inspires exaggerated respect and tempts the philosopher to bring the technique of theology to the help of the analytic method. God arrives to solve the difficulties of Berkeleian idealism or Bertrand Russell in less ambitious times invokes the Axiom of Reducibility.

In no branch of critical philosophy is this danger greater than in the analysis of mathematics, a discipline which acquires from its subject-matter a dangerous facility in the manufacture of vast systems of symbols whose architectonic

[1] For definition of the distinction between the philosophic and systematic aspects of any study cf. *infra*, p. 141.

complexity is occasionally of the same order as the labour required for their intelligent manipulation.

Recent research in the philosophy of mathematics has shown that each of the three principal theories of the nature of mathematics which are discussed in this book contains serious imperfections, some of which may be attributed to the causes indicated above. With this warning to the reader we may conclude these generalities and proceed to a preliminary summary of the three main types of theories which are to be the objects of our investigation.

Preliminary Survey of Three Types of Opinions Considered

Before commencing a detailed account a short description of the general features of the three main schools of mathematical philosophy with which we shall be concerned and their relations to one another may facilitate the orientation of the reader who is unfamiliar with the subject.

The three schools of thought chosen on account of their importance and influence are usually distinguished as Logistic, Formalistic, and Intuitionist, their best known living exponents being Bertrand Russell, David Hilbert, and L. E. J. Brouwer respectively. Their doctrines differ as much in methods of approaching problems as in their conclusions.

Logistic

The logistic thesis: pure mathematics is a branch of logic.

The programme of the logistic school has been expressed by Russell as follows: "Pure Mathematics is the class of all propositions of the form 'p implies q' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth" (*Principles of Mathematics*, p. 3). In other words, the propositions of mathematics are propositions of logic, they state relations between propositions whose content has been abstracted to leave only their form, shown by the logical constants *and*, *or*, etc.

On this view, all mathematical concepts such as *number*, *differential coefficient*, etc., must be capable of definition in terms of logical concepts, pure mathematics becomes a branch of logic and the distinction between the two subjects is merely one of practical convenience. Much of Russell's work, like that of his collaborator, Professor Whitehead, and his great predecessors, Frege and Peano, was devoted to performing the reduction of mathematical concepts to logical concepts. The culminating achievement of this school is Russell and Whitehead's *Principia Mathematica*, a massive work of bewildering complexity but great logical beauty, which purports to be a detailed reduction of the whole of pure mathematics to logic.

Formalism

The formalist thesis: pure mathematics is the science of the formal structure of symbols.

The formalists, on the other hand, deny that mathematical concepts can be reduced to logical concepts and assert that the many difficulties of logic which beset the path of the logistic philosophies have nothing to do with mathematics. They see in mathematics the science of the structure of objects. Numbers are the simplest structural properties of objects and are themselves objects with new properties. The mathematician can study the properties of objects only by making a system of signs which stand for them and by recognizing and allowing for the irrelevant features of the signs he uses. But provided he has an adequate system of signs he need no longer worry about their meaning since he can see in the signs themselves those structural properties which interest him. Hence the formalists emphasize the importance of the formal characteristics of the mathematician's sign-language, those which are independent of the meaning he may want to attach to them. This is not to say that mathematics is a meaningless game as the formalists have often been accused of asserting; they say that mathematics is concerned with the structural properties of symbols (and hence of all objects) independent of their meaning. This view has proved very fruitful in geometry and its success in that field is largely responsible for its widespread popularity. The formalists naturally lay a greater value upon a consistent symbolism than the logisticians; the contradictions in pure mathematics can be removed, they say, only by the provision of a symbolism which has been demonstrated to be foolproof. The demonstration itself cannot be carried through by the use of symbols independently of their meaning, for these symbols in turn would have to be legitimized and so *ad infinitum*; but they demand a demonstration using no process of thought essentially more complicated than that by which we see that two things and two things together make four. Most of the recent work of the formalists has been directed towards an elementary proof of the validity of mathematics from this angle. So far their success has been only partial, and there are grave doubts whether their programme can be consistently carried through.

Intuitionism

The intuitionist thesis: pure mathematics is founded on a basic intuition of the possibility of constructing an infinite series of numbers.

The formalists lay the emphasis on symbolism, the intuitionists on thought. For the latter the body of mathematical truth is not the timeless objective structure that it appears in the formalist and logistic philosophies. Mathematics, regarded as a body of knowledge, grows, it is a becoming, a process, which can never be completely symbolized—and even this manner of regarding it is perhaps dangerously abstract. Mathematics should be regarded as a social activity by which individuals organize phenomena in their most general aspect to satisfy their needs. Hence it is not enough to have a symbolism for mathematical thoughts; they are independent of the particular language used to express them. What is absolutely necessary is that the language should significantly express thoughts. We must be able to stop at every point in mathematics and see the state of affairs which is expressed as clearly as we can see the fact that to a heap of objects, no matter how many, it would always be possible to add one more and again one more in a never-ending process. Knowledge of this particular process, the possibility of indefinitely extending a series of objects by the addition of extra members, which may be expressed alternatively with sufficient precision for present purposes as direct knowledge of the sequence of the natural numbers, is termed the 'Urintuition' (basic intuition) by Brouwer; it is fundamental and irreducible in his philosophy.

His emphasis on the necessity for mathematical statements to have a clear 'intuitive' meaning leads him to reject general assertions such as "There is a prime number the sum of whose digits is divisible by 1004" on the ground that they are neither true nor false but meaningless. General statements have meaning, he asserts, only when a definite construction is known by which they might in theory (though not necessarily in practice) be tested for truth or falsehood with the certainty of obtaining an answer. So when and if a prime number is ever found the sum of whose digits is divisible by 1004 the assertion given above (or strictly the assertion which will then be expressed by the same words) will have sense. If general propositions whose truth can be tested by a known procedure be called *constructive* propositions it is easily seen that the contradictory of a constructive proposition is not in general constructive. This doctrine has often been misunderstood to amount to a denial of the law of the excluded middle that a proposition is either true or false.

Mutual Relations of the Three Schools

The logistic and formalist programmes have enormous difficulties to overcome if they are to be ultimately successful. For the logistic reduction of mathematics to logic breaks down at a crucial point and a complete formalist proof of the consistency of mathematics is probably impossible. But the intuitionist doctrines require the larger part of mathematics to be rewritten, reject proofs that have long been accepted, abandon large portions of pure mathematics, and introduce a disheartening and almost impracticable complexity into those domains which are remodelled.

The mutual interaction of the three movements are, briefly, as follows: the logistic thesis of the necessity for symbolizing mathematical proof has been completely adopted and improved in important technical aspects by the formalists, who use the logical notation evolved in essence by the logistic school. The intuitionists have, on the whole, been negatively influenced, reacting away from symbolism in consequence of the logistician's failures, but they too are beginning to produce an intuitionist formal logic. Research by the formalists, especially in geometry, has undermined the Kantian conception of space, and, by incidentally revealing the technical deficiencies in the logistic systems has largely destroyed what may be called the theological view of mathematics with its unrestrained belief in such transcendental entities as transfinite numbers. The influence of intuitionism has been very marked upon the other systems; it can be clearly seen in Hilbert's insistence on the need for finite non-formal proofs of the consistency of mathematics, i.e. what are now called metamathematical proofs, and also in modern demands for constructive development of such subjects as the theory of sets of points.

These three types of theories modify and inspire all the rest, but eclectic compromises are common. By using some of the innumerable modifications which a crowd of commentators and critics have devised it is possible and quite usual for the defenders of almost any philosophy of mathematics to shift their ground sufficiently to meet all criticisms. While drawing attention to such sophistries, we must not fall into the opposite extreme of judging philosophies of mathematics by their failures and omissions. We propose to judge them by their ability to analyse the whole field of mathematical fact and by the extent to which they can be formulated as precise and internally consistent systems. This is a test which requires a clearer statement of the opposing doctrines than their expositors have always provided, a test which none of the three philosophies here considered triumphantly satisfies.

SECTION I : LOGISTIC

SECTION I: LOGISTIC

This section will be devoted to a detailed description of a group of theories concerning the nature of mathematics which assert that mathematics should be considered as a branch of logic. If this opinion is correct the distinction between the two sciences, though venerable and established, is quite arbitrary. This claim is based on proofs which seek to demonstrate in detail how the reduction of mathematics to logic is accomplished.

Any philosophy of mathematics which includes this doctrine will for convenience of reference, and with the reader's permission, be qualified in this book by the adjective 'logistic'. This usage of the term is frequent in the literature of the subject and it is sufficient to mention another, less frequent, use of the same term, viz. as a substantive denoting "the science which deals with types of order as such" (C. I. Lewis, *Survey of Symbolic Logic*, p. 3), to forestall any confusion between the two meanings. The latter use of the word is based upon and implies a distinction between logistic, the science which treats of all types of order, and symbolic logic, that section of logistic which is concerned with the specific types of order exemplified by propositions; but our use of the word will not presuppose that this distinction is recognized by the philosophers whose theories will be termed logistic.

We commence with a brief historical summary of the views under consideration.

History of Logistic Views of Mathematics

A notice of the chief logistic writers from Leibniz to Wittgenstein.

The beginnings of logistic philosophies of mathematics are to be found in the gradual application to logic of a symbolic technique modelled upon the parallel use of symbols in mathematics. In its later stages this process was accompanied by extensive alterations in the traditional Aristotelian logic, by the introduction of many more propositional forms than Aristotle or those who expounded his logic recognized. This in time presented fresh opportunities for the application of symbolic technique, until finally systems of symbols were invented of sufficient generality to be used in the attempt to reduce mathematics to logic.

A convenient starting point for the present brief mention of the landmarks of this process of development is made by Leibniz, whose technical researches in symbolism preceded and often inspired the long series of inventors who perfected the algebra of logic. His work contained the germ of the entire logistic conception; it is no mere coincidence that many of the logistic philosophers find themselves sympathetic to Leibniz and inherit the characteristic atomism of his system.[1]

The significance for our purposes of Leibniz's studies in the algebra of logic [2] lies in the fact that no proof with any pretensions to rigour of the thesis that mathematics can be reduced to logic is possible without a well-developed symbolism and calculus for logic itself. Statements occurring in logic must be systematically symbolized in order that their relationships to mathematical theorems should become apparent. Leibniz, a mathematician of genius as well as a philosopher, was eminently fitted to begin the task of inventing the algebra of logic and his papers [2] show him to have made several attempts though with other motives.

Subsequent writers, of whom the most important are De Morgan (*Formal Logic*, 1847), George Boole (*An Investigation into the Laws of Thought*, 1854), E. Schröder (*Vorlesungen*

[1] The Axiom of Reducibility is a generalization of the Leibnizian principle of the identity of indiscernibles.
[2] Cf. C. I. Lewis, *op. cit.*, for further details.

über die Algebra der Logik, 1890–1905), and C. S. Peirce (see bibliography), by their elaboration of the algebra of logic fulfilled Leibniz's dream of a *Characteristica Universalis*, a calculus of reasoning suited for the logical analysis of concepts and the structure of scientific systems, and provided the necessary technical equipment for the logistic school. Schröder and Peirce emancipated symbolic logic not only from the Aristotelian view which permitted only the subject-predicate form for propositions but also to a great extent from the insistent preoccupation with mathematical analogies which retarded the early advance of the subject; the way is clear for the actual analysis of mathematics. The first important work of this second period was accomplished by R. Dedekind (*Was sind und was sollen die Zahlen?*, 1888), who supplied the now famous method of defining real numbers in the mathematical continuum in terms of the rational or fractional numbers. His work may be regarded as a continuation of Weierstrass's movement to 'arithmetize' mathematics, that is to reduce all pure mathematics to the study of the properties of integers; for after Dedekind the study of irrational numbers could be replaced by the study of certain classes of fractional numbers; and the reduction of the study of fractional numbers to that of integers presents no difficulties and had already been accomplished.

The definition of real numbers by 'Dedekind section' as his method is called, although accepted by mathematicians and used as the very foundation of the modern theory of functions, had to meet serious criticism which subsequently led to attempts at improvement by the logistic philosophers.

The next works of historical importance are Frege's *Begriffsschrift*, 1879, *Grundlagen der Arithmetik*, 1884, and *Grundgesetze der Arithmetik*, 1893–1903. The last two books completed the reduction of mathematics by defining the rational numbers in terms of logical entities. Unfortunately

Frege did not use Boole's calculus of logic, preferring an elaborate but clumsy symbolism of his own, whose intricacy prevented his work receiving the recognition it deserved; his books remained almost unknown until rediscovered by Russell after the latter's *Principles of Mathematics* had been written.

While Frege had given a philosophic analysis of the concept of number, the Italian mathematician Peano and his school (*Formulaire de Mathématiques*, 1895–1905), in the course of extensive researches in symbolic logic, had shown that all propositions concerning the natural numbers which are required in mathematics can be deduced from a set of five axioms.

The results of Dedekind, Frege, and Peano had covered in conjunction the whole field of elementary pure mathematics,[1] and by reducing the real numbers to integers, integers to entities occurring in logic, had supplied all the materials for the logistic thesis. There was still needed a synthesis to co-ordinate these results and remedy the imperfections of these early proofs. This was begun by Bertrand Russell in *Principles of Mathematics*, 1903, and continued in *Principia Mathematica* (first edition, 1910) written in collaboration with Alfred North Whitehead. These two books are at the apex of the second period in the logistic movement; they profess to prove, rigorously and with the utmost detail, the identity of mathematics and logic.

The first is a philosophical and polemical discussion of the logistic theories; the second, written except for a minimum of incidental explanation entirely in mathematical symbols, a proof of the theories.

Since *Principia Mathematica* little advance has been made by the logistic school and time has shown serious defects in that work, so that the third period has been one

[1] Excluding Cantor's theory of transfinite numbers at that time still undiscovered.

of successive attempts to consolidate a position which at one time Whitehead and Russell appeared to have reached triumphantly.

Among the most notable of these attempts are H. Weyl's *Das Kontinuum*, 1918; L. Chwistek's *Theory of Constructive Types*, 1923–5; and F. P. Ramsey's *Foundations of Mathematics*, 1927. All these defend a logistic position. In addition there remains the remarkable *Tractatus Logico—Philosophicus*, 1922, of L. Wittgenstein, a former pupil of Russell, whose conclusions, in many respects unfavourable to *Principia Mathematica*, should be regarded as the self-critical culmination of the logistic movement.

Tasks of a Philosophy of Mathematics

The finite and infinite problems of a philosophy of mathematics are the investigations of the notions 'integer' and 'continuum' respectively. The subsequent analysis tends to replace these unclear notions by more precise ones with the same formal properties. The plan of such analysis is outlined.

A philosophy of mathematics has two principal objects intimately connected with arithmetic and the theory of functions respectively:—

(1) To elucidate and analyze the notion of 'integer' or 'natural number',

(2) to elucidate the nature of the mathematical continuum. These are formidable tasks; ignorance of the correct answers has provided paradoxes which date back to Zeno.

For convenience of reference let these problems be called the finite and the infinite problems of mathematical philosophy respectively. They are distinct, although the solution of the second may presuppose knowledge of the solution of the first.

In spite of the contradictions which the second of these concepts appears to contain (p. 89), the notions of 'integer' and 'continuum' have been used with constant success and with such mutual agreement that the validity of proofs involving them can, with a few notable exceptions, be decided by the unanimous vote of those with sufficient mathematical training to understand them.[1]

It would therefore appear that the terms 'continuum' and 'integer' have meaning for the mathematician and the same meaning for all of them,[2] and the natural procedure for solving both the finite and the infinite problems would seem to be to examine as closely as possible, and subsequently to analyze, the meanings of these terms. Such an approach would be bound to emphasize the ideas which mathematicians associate with the symbols they use, rather than the apparent interconnection of these symbols shown by marks on paper. And the resulting analysis would need to be such as the mathematician himself could accept as clarifications of his notions. Similar remarks are applicable to the philosophic analysis of any system of interconnected notions. Such a programme has in effect been adopted by the so-called logico-analytic school of philosophers[3] who have, however,

[1] The principal exceptions are proofs involving transfinite numbers to which more detailed reference will be made later.

[2] This can scarcely be a truism; for the contrary view—viz., that mathematicians are discussing nothing and that their terms have no meaning—has been seriously discussed. Thus F. P. Ramsey in a paper read to the British Association (1926) said: "Mathematics proper is thus regarded [i.e. by the formalists] as a sort of game played with meaningless marks on paper rather like noughts and crosses; but besides this there will be another subject called metamathematics, which is not meaningless, but consists of real assertions about mathematics, telling us that this or that formula can or cannot be obtained from the axioms by the rules of deduction": (*vide* F. P. Ramsey, *Foundations of Mathematics*, p. 68). This is an inadequate account of the formalist philosophy of mathematics and it is extremely doubtful whether a theory of the meaninglessness of mathematics has ever had supporters in this crude form. Such a theory would find it hard to account for the agreement between mathematicians. If mathematics is merely a game played with symbols there is no reason except convention why the rules should not be broken; chess played backwards is still a game that can be played consistently, but a topsy-turvy mathematics would be false.

[3] These include G. E. Moore (*Philosophical Essays* and *Principia Ethica*), Russell (in *some* only of his writings, especially *Our Knowledge of the External World* and *The Analysis of Matter*), L. S. Stebbing

contributed but little to the analysis of mathematics, being rather concerned with the analysis of facts of everyday experience.

We will restrict ourselves to two comments on the scope of this method in the analysis of mathematical notions.

(1) In spite of the mentioned agreement between mathematicians, it seems possible to deduce contradictions from the mathematical notion of the continuum; these contradictions refer to the subject-matter of mathematics and can be deduced by formally correct mathematical reasoning (p. 89). They are sufficiently striking to have led a very celebrated living mathematician to speak of a vicious circle in present-day mathematics (Herman Weyl: "Der circulus vitiosus in der heutigen Begründung der Analysis" *Jahresbericht der Deutschen Mathematiker Vereinigung*, vol. xxviii, pp. 85–92, 1919). So it is not unfair to ascribe much of the agreement between mathematicians to the fact that they find no need to use in most proofs dubious notions such as "all properties of real numbers"; these occur nevertheless in mathematical textbooks and are an integral part of mathematics.

It follows that unless the result of philosophic investigation is to reveal that the contradictions in question are illusory, produced by fallacious reasoning, ambiguity of terms, or some other trivial cause, clarification of the notions used by mathematicians will be inadequate unless supplemented by revision; it will be necessary actually to alter the meanings attached by mathematicians to many terms and imperative to find new meanings so clear and consistent that the contradictions no longer occur. This is a process of analysis supplemented by synthesis. Such a procedure diverts emphasis

(*Introduction to Modern Logic*), J. Wisdom (*Interpretation and Analysis, etc.*), and J. Nicod (*The Foundations of Geometry*), all of whom sometimes and some of whom always emphasize that they are analyzing the *meanings* of words.

from the original notions to be analyzed, which in so far as they are confused and inconsistent permit of no exact analysis.

Much discussion has been devoted during recent years to proving that the contradictions are only apparent, that they are trivial confusions of no interest to mathematics. Welcome as such a conclusion would be to all except those philosophers whose lives have been spent creating philosophic systems based on the necessary existence of contradictions, these attempts have met with little success and the balance of critical opinion is against them.

Philosophic analysis of mathematical concepts therefore tends to become a synthetic, constructive process, providing new notions which are more precise and clearer than the old notions they replace, and so chosen that all true statements involving the concepts inside the mathematical system considered shall remain true when the new are substituted.

(2) Such constructive analysis may however acquire a purely formal character when instead of analyzing it replaces the concepts by a completely new set having the same interconnections. A process of this kind is appropriate in the analysis of mathematics whose 'formal' character we now proceed to examine.

Our conception of the nature of philosophic analysis as actually practised by the logistic school may be summarized in the following manner: the system to be analyzed contains a number of notions[1] denoted by symbols $a, b, c, \ldots$ say. These are combined in various theorems, say *abc*, *deab*, etc., which may be denoted by $A, B, C, \ldots$ From $A, B, C, \ldots$ taken together a contradiction can be deduced. Analysis attempts to replace $a, b, c, \ldots$ by new notions $a', b', c', \ldots$ say, so that as many of the corresponding theorems $a'b'c'$, $d'e'a'b'$, ... i.e. $A', B', C', \ldots$ may still be true, and

[1] 'Notions' is chosen as a neutral word and is not intended to prejudge the character of the entities which occur in the system. Thus $a, b, c, \ldots$ may include proper names, relations, adjectives, etc.

yet such that no contradiction can be deduced from $A', B', C', \ldots$ taken together.[1] And $a', b', c', \ldots$ may be either clarifications of $a, b, c, \ldots$ (genuine philosophic analysis) or merely any concepts of which the statements above are true (formal analysis).

[1] This is, of course, a very simplified account of the nature of a system omitting for example the distinction between the formal and non-formal elements of such a system.

Supplementary Note on Logical Analysis [1]

A discussion of the general features of all systems of symbols (languages introduces a definition of logical analysis in terms of the explained notions, multiplicity, significance, and structure. Difficulties arising in the logical analysis of language and here discussed throw light upon analogous difficulties in the analysis of mathematics.

Logical analysis is a method for elucidating the structure of systems of symbols or ' languages ', i.e. any set of symbols used in recurrent combinations for communication between persons. A language in this generalized sense always contains rules of syntax though not necessarily explicitly formulated. It will be convenient to confine the discussion to systems of symbols which constitute a language such as English, though much of what will be said is applicable to such systems as the languages of pure mathematics and physics.

Ambiguity of terms complicates the account of such systems, but this is unavoidable, since languages are made for use and not for analysis.

Logical analysis of a language is best understood in terms of the structure of the language. Though all readers of this account will be familiar with what is meant by structure it is not easy to give a short and, at the same time, accurate account of this notion. It may be described somewhat inadequately as the relations between the forms of complex symbols.

In more detail, what is meant by saying that language has a structure is essentially that certain elements (which are, as it were, the material out of which the language is built) recur as members in various complexes of elements while

[1] Extracted, with a few alterations, from the author's paper on "Philosophical Analysis", *Proceedings of the Aristotelian Society*, 1932–3, p. 237.

remaining recognizably the same in these different contexts, so that complexes can be transformed into one another by the reciprocal exchange of elements.

Complexes of symbols (phrases, sentences) can function as elements, and by substitution in other complexes lead to the construction of symbols of ever-increasing complication. To some extent this complexity is visibly manifested in the visibly-increasing number, variety, and arrangement of signs used, but to a great extent and for reasons of practical convenience this complexity is latent and is therefore revealed by the possible transformations of a given complex instead of by its *visible* complexity.

It is the purpose of logical analysis to make these complexities explicit by the discovery of laws for transforming symbols and by the manufacture of new symbols of sufficient visible complexity. It may be added that this would serve as a fair account of the mathematical method in general, and logical analysis is, in effect, a branch of applied mathematical investigation differing from what is conventionally known as pure mathematics chiefly in having a less abstract and more specialized subject-matter and from 'applied' mathematics only in dealing with linguistic elements in place of material bodies.

Logic proper is concerned principally with systems composed of words, and I must now particularize the foregoing account to apply to such languages. By *language* in the following paragraphs I shall usually mean the English language.

The elements of language which combine to form complexes include words, intonation, sentence-order, etc.

By elements are meant any features such as sounds, marks, shapes, etc., which can affect the senses of persons using a language for communication. In describing such elements it is necessary to distinguish those which are significant from those which are not. Significant elements in a sentence are those features whose variation alters the meaning of the sentence; thus in the printed sentence the word-order is significant, while the size of the letters composing the words is a merely accidental feature. It is not possible to make a very rigid distinction between the two kinds of elements; the spelling of words, though, in fact, a significant feature of printed sentences, is of merely conventional and trivial significance, for the spelling of all or any words in the English language might be simultaneously changed with no essential alteration in meaning.[1]

The definition of significance is in terms of *difference* of meaning, and this preliminary account of structure and significance must not be interpreted as an attempted definition of structure in terms of 'meaning', for the latter term is again subject to the peculiar ambiguity affecting all terms which have direct or indirect reference to mental processes. Though meaning is notoriously difficult to define, no final definition is needed for logical analysis, for whose purposes it is sufficient that some *distinctions* of meaning should be recognizable, for logical analysis is not a dissection of complexes into completely definite elements. Progressively more distinctions of meaning are perceived in the course of analysis, which is a process of successive approximations revealing increasing complexity of structure. The same is true of significant elements; it is impossible to enumerate in advance all significant features of a language, but the recognition of *some* such features is a sufficient starting point for analysis.

The differences in meaning with which I shall be concerned in this account are differences in *literal* meaning as distinct from metaphorical, æsthetic, or poetic meaning, for though the

[1] *Any* significant feature might be altered without injury to sense: cf. the Morse code, which employs only four elements (three are the theoretical minimum for any language), but such transformations do not destroy the structure of a language, which is what all the transformations of *that* language and no transformation of any other language have in common.

artist employs symbols they are insufficiently precise to lend themselves to logical analysis.

A list of the chief features of language significant with respect to literal meaning would include :—

(1) the occurrence of specific words or word-groups,

(2) word-order,

(3) emphasis,

(4) factual context,

each of which requires some explanation.

(1) It is a distinctive characteristic of all languages made to be spoken that groups of words combine into unities such as descriptive phrases, sentences, etc., which in turn can function like simple symbols and replace words in definite contexts. In most languages (in the widest sense) such groups are continually denoted by a single symbol concealing an underlying complexity of structure. Such substitutions, inevitable in the process of growth of any living language, are one of the circumstances which make logical analysis necessary. Limitations of the human larynx and the human memory demand the suppression of differences of structure which logical analysis has to reveal. That this is recognized to some extent in ordinary usage is illustrated by the fact that although difference in the marks or sounds used to express words is sufficient to, and usually does, indicate that the words are different, difference of words is based not on the difference of the marks which express them, but upon difference of meaning ; the same mark, if used with different meanings (e.g. *vice*, a carpenter's tool, and *vice*, for which sinners are punished), is said to express different words. This distinction must be rigidly preserved in the use of *words* and *symbols* in describing logical analysis, with the consequence that marks which would ordinarily be said to belong to the same word must be counted as belonging to *different* symbols ; thus the copula in *This is green* is not the same symbol as

that in *Green is a colour*, and both differ from the *is* in *A man is not a woman*.

(2) Word-order is a significant feature of language—for *Hitler hates Stalin* does not mean the same as *Stalin hates Hitler*—and plays a part in determining what combinations of words are nonsense. The latter term is to be taken in its strict meaning with none of the abusive connotations which render it so useful in philosophic discussion. Shorn of these it denotes simply any complex of symbols which is not constructed in accordance with the laws of combination (syntax) of the language in question. Examples of nonsensical combinations of symbols are such inadmissible groups as *succulent substantives*, *adjectives love analysis*, *the law of diminishing returns is blue*; these groups have no meaning as groups, and that fact is another aspect of the structure of language, for if *all* possible combinations of symbols were permitted the language would have a minimum or vanishing structure. Since our concern is with *literal* meaning, metaphorical or poetic phrases such as *yellow jealousy*, *necessity is the mother of invention* must also count as 'nonsense', though the latter type of nonsense differs from the former in being capable of being paraphrased into matter-of-fact language.

(3) Through preoccupation with the printed rather than the spoken word it is easily overlooked that intonation or emphasis is a significant feature of sentences. Shifting of emphasis from one word of a sentence to another usually alters the sense; increase of emphasis on one particular word alters what may be called the *intensity of emphasis* of the meaning.

The intensity with which a word is emphasized in a sentence corresponds to the degree of attention called to the use of *that* particular symbol with all its implications rather than any other. To emphasize a word is to state that only the word actually used will fit the situation and hence to imply, with varying degrees of definiteness, certain facts about the situation. If I say "*Mrs.* Jones did so and so", with a certain emphasis on the *Mrs.*, part of my assertion is roughly translatable as "It is Mrs. Jones, a married woman and no spinster, that I am referring to". In such a case I am not using *Mrs. Jones* as a (grammatical) proper name, but as a description; the two uses are quite distinct and the implications of two sentences in which they occur are very different.

It does not seem possible to remove the ambiguity often caused by doubt as to the degrees of absolute intensity of emphasis (of each word) and relative intensity of emphasis (of words in relation to one another) in a sentence by a convention that maximum intensity is in all cases to be employed, i.e. by a convention that *all* conceivable implications of any form of words are to be allowed. For this could not remove the difficulty of *relative* emphasis, and there is no maximum to the number of possible implications of the use of any symbol (except a logical proper name) in a sentence. The connotation of an attribute may include the existence of antecedently causal events which may, in turn, imply the previous existence of other events and so on—"being married," in one sense at least, entails having signed a book in the presence of a registrar, and "being a registrar" entails having been authorized by the proper authorities, etc.,—and *such* an infinite chain never *is* intended. Or, alternatively, at some stage, some 'simple' quality is attributed to some subject. In the latter case the common use of the same sign by various persons carries implications. To say so-and-so is red may (or may not) imply that the so-and-so has the colour *commonly* denoted by red, which in turn implies further statements. And language cannot be used so as to be deliberately charged with this kind of implication; for if by 'red' I mean "what is commonly denoted by red", then by 'commonly' I must mean "what is commonly denoted by *commonly*" so that either I can never express what I mean or else I am using language parrot fashion.

(4) The factual context of a sentence, i.e. the circumstances in which it is uttered or printed, serves in practice as a substitute for the direct symbolizing of structure and thus, by suppressing the manifestation of structure, may lead to confusions. It is not sufficiently appreciated that every form of words may express several different propositions according to context; this effect is well illustrated by considering the different meanings of *This is a white mantelpiece* as an answer to each of the following eight questions in turn:—

What is this white object ?
What colour is this mantelpiece ?
What is this object ?
This is not a white mantelpiece, is it ?
Is this a white or a black mantelpiece ?
Is this *the* white mantelpiece ?
Is *this* a white mantelpiece ?
Where is there a white mantelpiece ?

If the reader will take the trouble to repeat the sentence as if it were an answer to each of these questions successively the differences in literal meaning should soon become apparent.[1]

It may be objected that emphasis and intonation are subjective elements of language, indicating the attitude of a person asserting a proposition (or making a judgment) with respect to the order in which he wishes the terms to be considered, the relative importance he attaches to them, etc., and that there is a definite Oxford Dictionary meaning of any form of words even though the person using those words

[1] The number of variants is, of course, not confined to eight, and *some* could be more unambiguously expressed by the use of alternative forms of words.

is unaware of that full meaning. If that is a valid objection the task of logical analysis is considerably simplified, but it appears to be more correct to regard the significant sentence as being, as it were, two-dimensional, having both extent and intensity. The terms of which it is composed determine its extent or area of reference, the relative and absolute emphasis attached to its terms regulates the fashion in which the truth of the statement is tested.

This can also be expressed in another way; the significant sentence, i.e. a sentence actually in use to convey meaning, contains two heterogeneous elements in its expression: it names the members of a configuration of objects and indicates one of various possible correspondences between the sentence and the configuration. Thus the statement may categorically assert or deny, question, doubt, assert with varying degrees of probability, the existence of the configuration. This view may, perhaps, be made clearer by an example: I will assume that the reader knows that there is a cathedral on Ludgate Hill. It has often been said that in addition to the cathedral, and the hill (or better, perhaps, the cathedral-on-the-hill) there is also a *fact*, viz. *that there is a cathedral on Ludgate Hill*, and that it is the correspondence of this fact with the proposition, "There is a cathedral on Ludgate Hill" which makes the last a true statement. The alternative view here suggested is to consider the correspondence to be between two configurations of objects: (*a*) St. Paul's Cathedral with Ludgate Hill, etc., and (*b*) the symbols *cathedral, Ludgate Hill*, etc., in their arrangement in the proposition considered; and to regard the characteristic falling intonation with which the *is* is pronounced or understood to be pronounced as showing the *kind* of correspondence which is asserted. The correspondence is simple in its expression (the intonation which expresses it being comprehended as a simple symbolic feature like *red*, and not as a complex like *gold-fish*), but can be unfolded in a characteristic fashion by stating explicitly as many of the implications as there is time for on any given occasion.[1]

It is, however, possible to sketch the rudiments of logical analysis without taking into account the difficult questions associated with emphasis; this is in accordance with the general view of the nature of logical analysis explained above. For simplicity it is as well to break up the definition of logical analysis as follows :—

A is of the same type as B means: in every context where A can occur without making nonsense B can also do so, and vice versa. Here A, B are, of course, symbols, and it is easy to see that *being of the same type as* is a transitive symmetrical relation which separates all symbols into a set of mutually exclusive classes each containing all the symbols of the same type as any member of the class.

A is of the same level as B where A, B are propositional functions, that is symbols expressing qualities or relations,[2] means: all the arguments to A are of the same type as all the arguments to B. Propositional functions and their arguments are symbols.

I cannot define *propositional sentence*, and a description must suffice: Propositional sentences are a subclass of sentences, consisting of all those which express statements and are, therefore, neither questions, requests, or commands, and excluding all sentences which contain nonsensical combinations of symbols; tautologies, equations, identities, and contradictions may all be propositional sentences. *Sentence* will be used as an abbreviation for propositional sentence from this point onwards.

[1] The process of unfolding will, of course, not constitute part of the logical analysis of the sentence.

[2] Propositional functions are better defined as parts of propositional sentences obtained by omitting nouns or noun clauses; an argument to a propositional function is any word whose addition (with or without arguments) changes the propositional function into a propositional sentence.

A has the same multiplicity as B where *A*, *B* are sentences, means: the symbols composing *A* can be put into one to one correlation with the symbols composing *B* in such a manner that corresponding symbols are of the same type.

In applying the above definitions to investigate the multiplicities of specific symbols complications are produced by the systematic ambiguity of words which makes it difficult to recognize whether two symbols are of the same type; substitution of one for the other may *seem* to make sense because all the other symbols in the context are unconsciously replaced by new symbols of different type expressed by the same signs. Often, indeed, it is by no means easy to recognize whether two marks represent the same or different symbols, a circumstance responsible for many of the fallacies in philosophic reasoning.

It is therefore worth indicating how relations of identity and difference, whether of type, level, or multiplicity, can be recognized. Relations of multiplicity are internal relations between sentences, holding independently of the truth or falsehood of the assertions expressed by the sentences. They, and from them the corresponding relations of level and type, can be made more obvious by using the sentences *A* and *B*, say, under comparison as premisses in deductions. For if *A* and *B* have different multiplicities, but appear to have the same, then *some* deduction which will be correct when *A* is used as premiss will furnish a fallacious deduction when the deduction is transformed in such a manner that *B* takes the place of *A* and all else is unchanged. That is to say that since all logical deduction is in virtue of multiplicity of sentences, difference of multiplicity is revealed by the impossibility of reciprocal substitution in deductions.

Further, language usually provides alternative forms of expression for the same meaning; if *A* can be translated from one grammatical form into another, and *B* has the same multiplicity, it must be possible to translate B into a corresponding sentence.

So differences of multiplicity can be tested :—

(1) By performing the transformations mentioned in the definition of multiplicity.

(2) By translating A into equivalent sentences and performing the same transformations on B.

(3) By constructing deductions with A as a premiss and substituting B.

Finally then, *Logical Analysis of symbols consists of showing their logical form, that is their type, level, or multiplicity, more explicitly*. This can be done in several ways :—

(1) One symbol can be replaced by several, e.g. if a symbol A is found to have the same type as a group of two symbols B_1B_2 it must be possible to replace A by a group of two symbols A_1A_2 where A_1 has the same type as B_1 and A_2 the same as B_2; A_1A_2 means exactly the same as A, but their use leads to less confusion.

But (2) it is not possible to show explicitly *all* the multiplicity of a sentence in this fashion for the multiplicity is partly constituted by the fact that the sentence is composed of symbols of certain definite kinds and no others. Thus 'multiplicity' as here defined is not exhausted by the *number* of symbols which can be substituted and logical analysis will partly consist of statements *A is of the same type as B* where B is a symbol whose type is clearly understood and A is not. Or again, the same result can sometimes be achieved by statements of the kind : *A is a colour* or *A is a sense-datum* which indicate the type of A by describing the kind of context in which it can be sensically (or non-nonsensically) used.

Good examples of logical analysis are Russell's theory of descriptions, Moore's analysis of existential propositions, Wittgenstein's critique of identity.

It is now possible to define *logically misleading symbol*: If in some usage a symbol *A* occurs in a sentence which can be translated into another showing its multiplicity more explicitly, and such that *A* no longer appears in the new sentence, *A* is said to be a logically misleading expression in that usage.

Examples are :—

(1) *real* in *lions are real.*

(2) *fact* in *It is a fact that I work in the British Museum.*

For to write *lions are real* is to suggest that the sentence has the same multiplicity as *lions are fierce,* and this is not the case. The two *are*s are different symbols; *real* and *fierce* are of different types. *Lions are real* is better written *Something is characterized by being a lion* and *real* is therefore a logically misleading expression in that usage.

Again, *It is a fact that I work in the British Museum* can be more simply expressed by *I work in the British Museum,* and, therefore, *fact* in that usage is a logically misleading expression.

In elaborating logical analysis still further it would be necessary to distinguish between various kinds of logically misleading symbols. For the sense in which every condensed symbol such as *president* capable of being replaced by an explicit description is a logically misleading expression is not the same as that in which *fact* is logically misleading in some usages. The basis of the distinction is this: A logically misleading expression of the first kind can be replaced by a group of other symbols without alteration to the remainder of the sentence in which it occurs (e.g. uncle = brother of a parent), whereas a transformation of a logically misleading expression of the *second* kind involves alteration of other parts of the sentence as well (e.g. the transformation of *real* above). *Logical construction* is sometimes used by the logico-analyst

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in such a fashion as to be identical with a subspecies of the second kind of logically misleading symbol.[1]

The materials for the actual practice of logical analysis are partly available in the propositional calculus and calculus of relations elaborated by Frege, Schröder, Peirce Russell, and others, but the point of view of most of them differs fundamentally from my own, in neglecting such symbolic features as emphasis and in adopting an extensional view of symbolism, mistakenly thought to be necessary for the analysis of mathematics.

[1] The notions of multiplicity and type which have been used above were suggested by remarks made by Dr. Wittgenstein in his *Tractatus*, and also in lectures at Cambridge. Without making him responsible for my conception of analysis I think it will be found that my definition of multiplicity and logical analysis agrees in many respects with what he has said concerning analysis.

The Formal Character of Pure Mathematics

This section describes the ideal arrangement of a branch of pure mathematics as a system of deductions from initial axioms. It is a consequence of the generality of pure mathematics that the subject-matter of such a system is indefinite: the axioms treat of *any* set of objects whose names will fit into the axioms.

The theorems which constitute any branch of pure mathematics can be arranged in the following manner:—

First come a number of axioms containing those mathematical objects, such as integers, lines and points, groups, and their properties or relations, with which that branch of pure mathematics specifically deals. These axioms will usually take the form of general and existential statements concerning the properties and relations of the entities; the relations are named but the entities are referred to by indefinite descriptions.[1] Relations themselves can of course be the 'entities' of another system of axioms, and the theorems of one department may be the axioms of another. Axioms are so called because they are accepted without proof in the context of the branch of mathematics of which they are the axioms: they are the premisses from which all theorems, as distinct from axioms, are deduced. In what follows 'theorems' will be understood to exclude axioms.

The objects referred to by indefinite descriptions in the axioms, together with their properties and relations, may be called the subject-matter of the particular system of inter-connected theorems in whose axioms they occur; their mutual relationships are specified by the axioms and thereby determine the character of all the theorems which follow.

[1] More accurately, using terminology defined later, symbols denoting 'entities' appear as apparent variables, symbols denoting their properties or relations as undetermined constants.

Thus geometry will be characterized by axioms dealing with lines, points, etc., the theory of groups by axioms in which groups are mentioned. Arithmetic is in a peculiar position since definite integers occur in all systems of axioms, but even that subject can be arranged as above to begin with axioms whose subject-matter consists of integers and relations between integers.

In each branch of mathematics considerable choice can be exercised in selecting axioms, for many alternative sets can be obtained by suitable arrangement of the fundamental objects, but this fact is of minor importance for the present discussion.

Theorems are obtained by logical deduction from the axioms, which implies that no objects must be mentioned except entities composing the subject-matter nor any statements concerning them except the axioms. For the purposes of mathematics all that needs to be known of these objects is stated in the axioms and this is true not only for the subject-matter of a given branch of mathematics, but of all objects which occur in mathematics since, by combination, a set of axioms could be constructed for the whole of mathematics.

It follows that many different sets of objects and relations can serve as the subject-matter of any given mathematical theory. For example, the 'points', 'lines', 'circles', etc., which are the subject-matter of the axioms of Euclidean geometry are primarily understood to be the geometrical figures usually denoted by these names; yet the axioms remain true if the following transformations are made: 'points' are taken to mean ordered [1] triplets of real numbers, 'lines' are translated into linear equations in three variables, and statements such as "the point P lies on the line l" into the statement that the corresponding triplet of real numbers

[1] So that (1, 5, 6) say is not the same 'point' as (5, 1, 6).

satisfies the corresponding linear equation, etc.[1] Since axioms of pure geometry can be translated in this fashion all theorems deduced from them undergo the corresponding transformation; all statements of pure geometry may be interpreted either as concerned with points, lines, circles, etc., in the common significance of these terms, or with certain sets of numbers, equations, etc. The theorems of pure mathematics are true of any objects and relations which satisfy the axioms[2]; and transformations of meaning of the type described can be performed in any branch of mathematics which can be arranged in the form of axioms and theorems.

Hence, although even mathematicians themselves associate such terms as 'line' and 'point' with images of definite geometrical figures, the names function as terms of variable meaning whose use facilitates the construction of very general theories of the relations between many different systems of objects and exhibits the common structure of these various systems elegantly and succinctly.

The formal character of pure mathematics described in the immediately preceding paragraphs indicates why an 'analysis' which substitutes new notions for the notions to be analysed is a legitimate process. Any analyses of mathematical terms which left the mathematical theorems superficially unchanged must not be summarily rejected on the ground that they are repugnant to common sense or that they are not analyses of mathematicians' notions.

This apology for formal analysis requires two important reservations in the case of pure mathematics. (1) The natural numbers as we have just seen are in the peculiar position of

[1] The fact that this transformation is possible is the basis of Cartesian or co-ordinate geometry which is essentially the application of algebraic methods to geometry by transformations of the type sketched in the text. For further details cf. D. Hilbert, *Grundlagen der Geometrie*.

[2] Every system of things will have *some* relations and will therefore satisfy some conceivable system of axioms, so every system of things will have a geometry; mathematics studies the more 'interesting' of these.

occurring as constants in all axiom systems and therefore marks denoting integers must be understood in a sense in which lines, points, etc. need not be understood. (2) No complete axiom system can be set up for 'real numbers'. That is to say in the two cases where the fundamental problems of philosophical analysis of mathematics arise it will be found that no 'formal' analysis is adequate. A justification of this thesis however requires further explanation of the nature of axiom systems and will be reserved for a later section.

The next topic for discussion is the so-called propositional calculus, the elementary portion of the algebra of logic.

The Propositional Calculus

The manipulation of propositions, definitions of *implication, equivalence, tautology*, and a typical proof.

We turn now to the details of the logistic proof of the identity of mathematics and logic; until further notice the system considered will be that of *Principia Mathematica*, first edition, but we have substantially revised the account of the matter to be found in that book and made considerable use of improvements that have been perfected since its appearance. We begin by a short account of the post-Aristotelian view of the nature of logic and of the manner in which an algebra of logic is constructed.

Logic deals with such relations between propositions as depend only on the logical form of the propositions and not on their content. In order to explain what is meant by logical form it is best to begin with an illustration; the two propositions *the sky is blue* and *the grass is green* have the same logical form, for if *the sky* is substituted for *the grass* and *blue* for *green* the one proposition transforms into the other. It is difficult to give an exact description of *logical form*; the following is a good approximation: the logical form of a proposition is that which it has in common with all propositions whose constituents can be put into one-one correspondence with its own constituents. But the notion of constituent is not sufficiently precise for this definition to be satisfactory. What is desired is that words like *red, house, Jones,* when occurring in a proposition, should denote constituents of the proposition, and that words like *is, not,* *or*, should not.[1] The difficulty of defining logical form will not affect the exposition of the logistic calculus of propositions where it is never necessary to mention *parts* of propositions, but it is important later in the calculus of propositional functions.

Formal logic studies the rules which state the conditions under which the truth of a proposition, p say, can be deduced from the truth of a set of propositions, $p_1, p_2, \ldots p_n$ say, by virtue of their logical form alone. The classical syllogistic rules will illustrate this; for they state the circumstances in which a proposition can be deduced from two others. We call relations which hold between propositions by reason of their logical form internal relations. Among the most obvious kinds of internal relations between propositions are those between compound propositions such as *this paper is white and this line has several words* and simpler propositions like *this paper is white* which are part of them. If the logical forms of propositions are known, deductions can be made from them without reference to the particular state of affairs they assert. Thus, if *this paper is white and this line has several words* is true, the truth of *this line has several words* can be deduced in consequence of the relation between the forms of the two propositions, the particular assertions they make being irrelevant; and in general if a proposition *p and q*, where p and q are any propositions whatsoever, is true the truth of p (and also of q) can be deduced.

The appropriate symbolism for all statements of how propositions can be deduced from other propositions by

[1] This description of logical form implies a conception of logic which would be inacceptable to some logicians: those who agree, with Aristotle, that all propositions have the subject-predicate form would say that the definition in the text supplies too many 'logical forms'; Wittgenstein (in *Tractatus Logico-Philosophicus*), on the other hand, needs more forms than the definition supplies. For the definition of 'multiplicity' implied though not actually stated in the *Tractatus* makes logical multiplicity a narrower notion than logical form defined above. Two propositions of same multiplicity must have the same logical form, but the converse is not always true.

reason of their logical form, together with the appropriate rules for manipulating such statements in order to obtain others, is called the propositional calculus. The symbolism is such that propositions are always represented either by simple variable symbols such as p, q, r, or by complex symbols which consist of the simple symbols connected by a small number of words such as *not*, *or*, *and*, which indicate logical form. All considerations of internal relations between propositions which involve reference to their constituents are reserved for the calculus of propositional functions.

For convenience of manipulation *not-p* or *p is false* is written $\sim p$; *p or q* is replaced by $p \vee q$, *p and q* by $p \,.\, q$. The word *or* is ambiguous; the meaning chosen is such that the assertion of $p \vee q$ does not exclude the possibility that both p and q are true. It is also necessary to symbolize the relation which holds between two propositions p and q when the second can be deduced from the first; this is expressed by saying *p implies q* or, in symbols, $p \supset q$. If however the word *implies* is used with this meaning it is found very difficult to develop a calculus; therefore a modified definition is adopted and $p \supset q$ is understood to mean "either p is false or q is true" which is equivalent to "it is false that p is true and q false".

The relation *implies* is therefore not an internal one, as is shown by the fact that it holds between any false proposition p and any true proposition q, irrespective of their logical forms.[1] This fact is without detriment to the use of the calculus since we need in actual deduction to deduce propositions only from propositions already known to be true; and it follows at once from the definition of *implies* that, if $p \supset q$ and p are both true, then q must necessarily be true. Although $p \supset q$ will be asserted in some cases where there is no internal

[1] The internal relation which corresponds to the first definition of *implies* is usually referred to as the *entailing* relation. This terminology is due to G. E. Moore (*Philosophical Studies*).

relation between p and q, all cases where there is the corresponding internal relation can consistently be represented by $p \supset q$; and the use of this symbol will lead to no mistakes as to the truth of propositions.

Some further definitions are required: Two propositions which imply one another are said to be equivalent and the statement p is equivalent to q symbolized by $p \equiv q$; the use of this sign to denote equivalence must not be confused with the use of '$= \ldots . Df.$' which means 'equals by definition' and is used for defining the meaning of new symbols in terms of those already known. For example the verbal definition which has just been given of *equivalence* in terms of implication can be expressed as follows:—

$$(p \equiv q) = \{(p \supset q) . (q \supset p)\} \ Df.$$

Here, as in general, the *definiendum* is placed to the left and the *definiens* to the right of the sign of equality, while the occurrence of the symbol $Df.$ signifies that the expression preceding it is a definition and not a theorem of the propositional calculus.

The preceding definition indicates the necessity for using brackets in order to render unambiguous the meaning of complicated expressions. *Principia Mathematica* adopts an ingenious method, replacing the conventional pairs of enclosing marks used for bracketing mathematical expressions by groups of dots . : :. :: etc. Each complete group of dots functions as a bracketing mark with the convention that any group of bracket dots dominates a group containing fewer dots. Thus the expression $p . \vee :: q . \vee :. r : p \vee q :.$ would be expressed in the older notation by

$$p \vee (q \vee (r . (p \vee q))).$$

The use of dots for brackets cannot be confused with the use of dots to symbolize the logical constant *and* (*supra*), for *and* always occurs between two complexes of signs which denote propositions, while bracket dots cannot do so. If suitable conventions are formulated as to the relative strength with which the signs . $\vee$ $\supset$ $\equiv$ and $\sim$ bind propositions the number of bracket dots required to symbolize a given expression can be considerably reduced; just as in the algebra of integers $(a \times b) - (c \times d)$ can without risk of ambiguity be written $ab - cd$, the expression $pq \vee rs$ in the algebra of propositions is understood to mean

$$(p.q) \vee (r.s).$$

Such simplification has only partially been performed in *Principia Mathematica*.

The materials of the propositional calculus having now been sufficiently enumerated it remains to explain how logical calculations are performed The purpose of the calculus is to determine which formulæ composed of symbols for variable propositions and logical constants remain correct for all determinations of the variables; an example of such a formula is $p.p \supset q : \supset .q$ (*if p is true, and p implies q, q is true*). Such formulæ will be called *tautologies*.

In accordance with the usual procedure of pure mathematics the calculus commences with a number of axioms, formulæ which must be seen without proof to be tautologies. These axioms of the propositional calculus were called 'primitive propositions' in *Principia Mathematica*. Further tautologies, the theorems of this calculus, are derived by the use of specified rules of manipulation described below.

The advantages to be derived from the use of a propositional calculus of this type are those inherent in the mathematical method. By indicating at all crucial points of a complicated demonstration the axioms, previously proved theorems, or manipulative principles, which are used, it becomes possible to test each step of a proof and to be certain that no fallacious reasoning has been introduced. This aim is of particular importance in the logistic thesis which requires scrupulous care to ensure the absence of all extra-logical elements.

In any branch of pure mathematics the rules of manipulation used in deriving theorems from axioms include the principles of logic; in the special case of the propositional calculus which is used to prove logical principles some logical principles occur twice, as formulæ and as principles for manipulating formulæ to obtain tautologies; this may be compared with the dual occurrence of integers in an axiom-system of relations between integers.

The principles of manipulation used in the propositional calculus of *Principia Mathematica* are the following :—

(1) The principle of substitution: tautologies are obtained whenever some propositional symbol, p say, is replaced ***wherever it occurs*** in a given tautology by some other one and the same propositional symbol. An example: by replacing p in the tautology $p \vee \sim p$ (p is either true or false) by $p \vee \sim p$ the following tautology is obtained :—

$$(p \vee \sim p) \vee \sim (p \vee \sim p),$$

(2) the syllogistic principle: if both A and $A \supset B$ have been shown to be tautologies, B is a tautology. Here A and B can be any formulæ.

This account of the propositional calculus may be concluded by a specimen proof, typical of others in the calculus. To assist the reader ordinary brackets have been used. We begin with the primitive propositions used in *Principia Mathematica*, viz. :—

(1) $(p \vee p) \supset p$

(2) $q \supset (p \vee q)$

(3) $(p \vee q) \supset (q \vee p)$

(4) $(p \vee (q \vee r)) \supset (q \vee (p \vee r))$

(5) $(q \supset r) \supset ((p \vee q) \supset (p \vee r))$

It is required to demonstrate that the expression

$$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$$

is a tautology. According to definition

$$(p \supset q) = (\sim p \vee q)\ Df.$$

Substitution of $\sim p$ for p and $\sim q$ for q in the fourth of the primitive propositions quoted above and replacing $\supset$ by its definition supplies the tautology

$$((\sim p) \vee (\sim q \vee r)) \supset ((\sim q) \vee (\sim p \vee r))$$

which, by the same definition supplies

$$(p \supset (q \supset r)) \supset (q \supset (p \supset r)). \qquad (a)$$

Substitution of $\sim p$ for p in the fifth of the axioms and use of the definition for $\supset$ furnishes in similar fashion the tautology

$$(q \supset r) \supset ((p \supset q) \supset (p \supset r)). \qquad (b)$$

Substitution of $q \supset r$ for p, $p \supset q$ for q, $p \supset r$ for r in (a) provides a tautology

$$((q \supset r) \supset ((p \supset q) \supset (p \supset r))) \supset ((p \supset q) \supset ((q \supset r) \supset (p \supset r))) \qquad (c)$$

which is of the type $A \supset B$ where A is identical with (b) already shown to be a tautology. The second principle of manipulation permits the deduction that B, viz. the expression $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$, is a tautology as was required to be proved.

This completes the account of the propositional calculus of *Principia Mathematica*; comment, criticism, and the consideration of possible modifications may profitably be reserved until the calculus of propositional functions has been described.[1]

[1] For further detail the reader must be referred to *Principia Mathematica*, to Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* or to Carnap's *Abriss der Logistik*.

The Calculus of Propositional Functions

Russell's definition of propositional function described.

We have seen that the purpose of a logical calculus is to symbolize as completely as possible the logical form of propositions and the internal relations which hold between propositions in consequence of their logical form, and to provide rules for the demonstration of tautological formulæ. The propositional calculus partially satisfies these demands but is not able to symbolize the logical form of such propositions as cannot be analyzed into conjunctive or disjunctive combinations of simpler propositions. Consider for example the following tautology: *If A is B and B is C then A is C*, whose tautological form is ensured by the logical form of the simpler propositions which enter into its composition. The resources of the propositional calculus will only suffice to indicate that *A is B*, *B is C*, *C is A*, are different propositions, but cannot indicate that relation between the structures of the three propositions which allows the third to be deduced from the logical product of the first two.

The extra symbolic machinery required is furnished by Russell's 'propositional functions' which he defines as follows: "A propositional function is simply an expression containing an undetermined constituent, or several undetermined constituents, and becoming a proposition as soon as the undetermined constituents are determined. If I say 'x is a man' or 'n is a number' that is a propositional function" (*The Monist*, 1919, p. 162), and again: "Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determinate meaning. Then ϕx is called a propositional function" (*Principia Mathematica*, vol. i, p. 15). It is easy to see the connection between *logical form* and *propositional function* as so defined. For it has been seen that the form of a proposition is what it has in common with all propositions whose constituents can be put into an ordered one to one correspondence with its constituents. Hence, if some of the words occurring in the proposition be replaced by symbols such as *x*, *y*, attention will be explicitly drawn to the form of the proposition rather than its meaning, and the symbolic construct so obtained will serve to define the propositional form without reference to a specific proposition.[1]

Propositional functions were independently used by Frege and are a distinctive feature of logistic systems; they were introduced by analogy with mathematical functions and are used in conjunction with the mathematical terms 'variable' and 'value', which, in common with 'function' itself, unfortunately have very ambiguous meanings in mathematics.

The consequence in *Principia Mathematica* is a lack of clarity as to the meaning of propositional function which has done much to confuse its readers. A short discussion of the mathematical notions of *variable, value of a variable,* and *function* will therefore be advisable.

[1] Cf. Russell, "Philosophy of Logical Atomism," *Monist*, 1919, p. 202. "I mean by the form of a proposition that which you get when for every single one of its constituents you substitute a variable." This, however, is not quite correct, since it would imply that the form of propositions is a variable propositional function; the correct view is that the form is what the proposition has in common with the variable propositional functions derived from it by changing all its constituents into variables.

Variable and Function in Mathematics

The purpose of this section is to give definitions of the terms variable and function as used in mathematics and to distinguish between the various usages in which they occur.

In what follows we shall often have to speak of symbols, using a word of great ambiguity which might conceivably lead to confusion. Without attempting to analyze or describe the meaning of the term *symbol* it may prevent some of these confusions to observe that, in the sense intended, *symbol* is a word of the same logical type as *word*. Anything that can significantly be asserted of a word can be significantly asserted of a symbol, and vice versa; symbols include words and algebraic signs such as x, y. The relation of the symbol x to the mark or sound which is used to express it is the same as the relation of a word to the mark or sound which expresses the word.[1]

A symbol is said to be a variable in mathematics if it is used to denote any one of a certain set of mathematical objects; *which* of these objects it denotes being left completely indeterminate.[2] The totality of these objects may be called the field of variation of the variable. The usefulness of a variable symbol in mathematics is due to and is exhausted by its ability to denote a member of its field of variation without an inconveniently exact specification of that member.

The values a variable can assume, or, elliptically, the possible values of a variable, are the objects contained in its field of variation. An example: if x is a variable real number

[1] This is, of course, a very sketchy account of the relation between symbols, words, and signs.
[2] The 'objects' themselves may in turn be variables.

between 0 and 1 its possible values consist of the numbers 0, 1 and all the real numbers which lie between those limits.

The signs chosen for variables are usually taken from the end of the alphabet, e.g. x, y, z. In accordance with what has been said, variable symbols will be particularly useful in all cases where statements are to be made which apply indiscriminately either to *any* member or to *all* members of a certain totality of objects. From this primary use of the signs x, y, z, etc., are derived various others which as they are liable to be confused with it must be considered separately.

Various Usages of Variable Symbols

New definitions, often used in the sequel, of the illustrative, formal, determinative, and apparent uses of a variable symbol.

(*a*) A variable symbol may sometimes be used to denote a member of its field of variation in a theorem or proof when some particular member must be chosen but any member of the field of variation is equally suitable. A statement containing the variable in this usage illustrates relationships which hold no matter which member of its field of variation the variable denotes. The statement $x \times y = y \times x$ in elementary algebra is a good example. $x \times y = y \times x$ illustrates all the relationships $2 \times 3 = 3 \times 2$, $4 \times 9 = 9 \times 4$, etc. This will be called the *illustrative* use of the variable sign.

(*b*) A variable symbol may occur as part of a larger construct partly or wholly in order to indicate formal features of that construct. The most important example of this use is the occurrence of variable signs as arguments to a function, e.g. x in ϕx (in statements containing ϕx as grammatical subject). This use is quite distinct from (*a*); x no longer denotes a member of its field of variation but is used to complete the sign of which ϕ is a part and to indicate that the propositional function ϕ takes one argument. Or again the symbol $\hat{x}$ may be used in conjunction with the name of a function to show that it is the function which is being discussed rather than one of its values.

The use of a variable to indicate formal properties of symbol constructs of which it forms part will be called its *formal use.* Any symbol, not necessarily variable, indicates more or less explicitly the form of any larger symbol of which it forms part, but variables are often explicitly used in order to draw attention to the form : cf. the example above : *If A is B and B is C then A is C.* Here *A*, *B*, *C* are variables which occur primarily in the formal and not in the illustrative usage for their field of variation remains completely indefinite and unspecified. A symbol may of course occur in several usages simultaneously.

(*c*) A variable may be used to denote a mathematical object known by a description insufficient to determine it exactly. In such a case the field of variation of the variable consists of all the objects to which the description in question applies and variable symbols are often used in this manner in order to determine these objects more exactly. This usage will be called the *determinative.* In a particularly important special case, viz. *reductio ad absurdum* proofs, the variable is used determinatively to denote a member of a field of variation which proves to be empty.

(*d*) A variable may occur in expressions which denote the result of mathematical operations on its field of variation. $\int_0^1 x^2dx$ denotes the result of performing in succession the two operations of squaring and integrating over the range of variation of the variable consisting in this case of the real numbers between 0 and 1.[1]

In such cases the variable is no longer capable of further

[1] As part of x^2 the variable also occurs in its formal usage.

determinations, the symbol of which it forms a part is a constant, and in such usage the variable is usually called *apparent*. This terminology however appears to have been invented for the logical calculus and does not occur in mathematics. The variable is termed *apparent* as opposed to *real* because it is no longer capable of varying, i.e. of being replaced by a symbol denoting a member of its field of variation. To summarize :—

in its illustrative use (*a*) the variable indicates an indeterminate member of a *known* field of variation ;

in its formal use (*b*) the variable indicates certain formal characteristics of larger symbols in which it occurs ; exact knowledge of the field of variation is usually irrelevant, emphasis being laid on the mere possibility of the 'variation' of a variable symbol ;

in its determinative use (*c*) the variable is used to obtain a more exact description of its field of variation ;

and, finally, in its apparent use (*d*) the variable occurs as parts of symbols denoting constants obtained as the result of operations on the field of variation of the variable.[1]

Definitions of Mathematical Functions

Two sharply contrasted definitions of mathematical function are current ; in the extensional definition a function is an extended list of pairs of numbers ; the intensional definition is in terms of the relation part and whole between symbols.

After this digression we may return to the definition of mathematical function. When the values of a variable, *y* say, are connected with the values of another variable, *x* say, in such a manner that whenever a value of *x* is known

[1] The first three definitions are new ; the distinctions they are based on were partially recognized in *Principia Mathematica* by the use of the thoroughly confused 'cap' notation for variables.

the corresponding value of y can be determined, y is said *to depend on x*, or y is said to be *a function of x*, or there is said to be a *functional relation* between x and y. These alternative phrasings correspond to various ways of regarding such a situation.

The case considered in the above definition is a particularly simple one, that of a *one-valued function of a single variable*; if several values of y correspond to each value of x, y is said to be a two-, three-, . . . valued function; if the knowledge of the values of other variables as well as the values of x is required in order to determine the values of y the function would be one of several variables. But no essential difference is produced by this additional complexity.

It has already been seen that the purpose of introducing variable symbols in all their various usages is to be able to make statements concerning their fields of variation, varying emphasis being laid either on the variable or its field of variation according to the purpose for which the variable is being used. The concept of function derives from that of variable, being the generalized notion of the interdependence of variables. It will be convenient and not misleading to restrict the discussion to the case of the one-valued function of one variable, that is to the case in which two fields of variation are connected in the particularly simple fashion described above.

Let us now proceed to consider in greater detail various methods of regarding the functional relation. First, the so-called *extensional* conception of the nature of mathematical functions.

When it becomes necessary in the course of mathematical proofs to consider a relation between the members of two classes of objects it is sometimes natural to define the correspondence in question by enumerating the pairs of corresponding objects in the two classes and to consider the functional relation in such a case as a correspondence between two fields of variation, that is between two collections of objects, rather than as a relation between two variables (symbols). Such is the case when each field of variation consists of a finite number of known objects. In the extensional conception of mathematical functions a one-valued function is always conceived of as a many-one correlation between two collections of objects, the correlation being defined by a complete list of pairs of corresponding objects. These pairs must be ordered in such a way as to indicate which objects belong to the same collection; an obvious convention is so to write the list that the left-hand members of each pair belong to one collection, while the right-hand members belong to the other. On this view a function is identical with such a list.

This is the view of mathematical function current in present-day pure mathematics. The same conception is however extended with doubtful justification to include cases when the fields of variation in question have an infinity of members, as happens, for example, if one field of variation is that of the real variable and consists of all real numbers. Mathematicians often regard functions as lists of infinitely many pairs of numbers.

Such a definition of a function as equivalent to a collection of ordered pairs of objects makes no mention of, and does not appear to involve, the notion of a variable; and the distinction between the dependent and independent variables, which recurs so often in mathematics, would appear purely arbitrary and unnecessary if imposed upon such a definition. When however one of the two collections of objects is infinite (as in the definition of a function of a real variable) it becomes impossible actually to write out the list of objects correlated; and although this fact does not distress the pure mathematician who contrives to think of his infinite collections of ordered pairs *as if* they were set out for inspection [1] it is at this point that the extensional notion of function is seen to be inadequate and needs to be supplemented by the notion of a function as expressing a *law* of correlation of two variables.

The alternative view of the nature of mathematical functions regards the use of variables as fundamental and provides a new definition which differs in many respects from the extensional definition. For if the values of two variables, x and y, are to be connected without recourse to the enumeration of pairs of corresponding values, this can be accomplished only by a law indicating how from *any* given value of x, no matter which, the corresponding value of y can be calculated.

[1] The attitude referred to can be well illustrated by quotations from F. P. Ramsey's *Foundations of Mathematics*, e.g. "It is obvious that two classes could be similar, i.e. capable of being correlated, without there being any relation actually correlating them" and again, "Real numbers are defined as segments of rationals; any segment of rational is a real number. . . . *It is not necessary that the segment should be defined by any property or predicate of its members in any ordinary sense of predicate.* A real number is therefore an extension and it may even be an extension with no corresponding intension. In the same way a function of a real variable is a relation in extension, *which need not be given any real relation or formula*" [p. 15—italics inserted]. These are however statements of a very extreme position which would probably be qualified by most mathematicians. Thus, e.g. E. W. Hobson, *Theory of Functions of a Real Variable*, vol. i (ed. 3), p. 272, defines 'the functional relation' as follows: "If to each point of the domain [or field of variation] of the independent variable x there be made to correspond in any manner a definite number, so that all such numbers form a new aggregate which can be regarded as the domain, or field, of a new variable y, this variable y is said to be a [single-valued] function of x." And although he proceeds to say: "In this definition no restriction is made *a priori* as regards the mode in which corresponding to each value of x, the value of y is assigned; *and the conception of function contains nothing more than the notion of determinate correspondence in its abstract form, free from any implication as to the mode of specification of such correspondence*." [my italics] he immediately adds: "In any particular case, however, the special functional relation must be assigned by means of a set of prescribed rules or specifications," and later explicitly excludes the case of an infinite table of values: "It is sometimes said, in order to illustrate the generality of the functional relation, that a function is definable in the form of a table which specifies values of y corresponding to the values of x. The inadequacy of such an illustration is manifest, if we consider that, even if the table were an endless one . . . no aggregate of y values can be defined by an endless set of numbers apart from the production of a norm [or law] by which these numbers are defined" (p. 274). Insertions made in the above quotation are shown in square brackets.

If both variables are taken in their illustrative usage this connection can be neatly indicated by an equation involving the two variables. From each such equation a law of calculation can be extracted; for example the equation $y = \sin x$, where both variables occur in the illustrative usage, will define a certain function of x, the law of calculation implied being that the value which corresponds to any given value of x is obtained on finding the sine of x by evaluating the appropriate convergent series.

This conception may be crystallized into an *intensional* definition of function which should be contrasted with the previous extensional definition. A *symbol* is now said to be a function of a second symbol if it contains the second symbol as part of itself, e.g. the symbol x^2 is a function of the symbol x.

There are several important points to be noticed concerning this definition:—

(*a*) Although applicable to any kinds of symbols, the definition is designed for use only when the second symbol is a variable, say x. x is then said to be an argument to the function.

(*b*) A function may have several arguments, that is several different parts containing variable symbols, but for simplicity we shall assume as before that this is not the case and that only one argument occurs.

(*c*) If x is replaced in the function by one of its values, the resultant expression then becomes a constant which is said to be *the value of the function for that value of the argument.*

(*d*) In comparing the extensional and intensional definitions it will be seen that while the former must refer to the 'mathematical objects' denoted by the variables, the latter is defined in terms of symbols alone. If it is considered that the correct definition of mathematical function should have reference to the objects denoted by the symbols it is easy to modify the intensional definition here adopted. The form we have chosen emphasizes that the intensional notion of function is always based on the use of *variable, argument*, etc., concepts which cannot be made precise except by reference to systems of symbols, but need not involve explicitly the concept of *object denoted*.

(*e*) The term *function*, in a meaning derived from that given by the intensional definition above, comes to be used for the *manner* in which the symbol, which is the function of *x*, is formed out of *x*, i.e. for the *form* of the symbol which has previously been called the function of *x*. This is the sense in which the mathematician will speak of 'the sine function' or 'the logarithmic function', meaning neither a symbol nor a correlation but the manner in which the function-symbol is related to its argument or the manner in which the corresponding values of the two variables are correlated, i.e. the form of the rule which establishes the correlation. This conception is particularly important for the mathematician: in the 'theory of functions of a real variable' it is precisely generalized properties of this kind of function which he is, for the most part, engaged in studying. And it is clear that this notion cannot be derived from the extensional definition of function, for the only possible abstraction to be derived from a bare collection of pairs of values where no law of correlation is assumed is the general notion of such collections.

(*f*) An extensional definition of a functional relation between two variables is only possible when the members of both fields of variation are known and can be enumerated. Not only is this impossible, as already stated, when either field of variation has an infinite number of members, it is also impossible to form extensional functions of a variable *x* occurring in the determinative usage. For in that case the members of *x*'s field of variation are unknown (and may not exist) so that their enumeration is impossible. It is however still possible to form intensional functions of variables in this usage, since the fields of variation are relevant to functions as intensionally defined only to the extent that they restrict the types of the resulting functions; symbols involving x as part can be constructed without knowledge of x's field of variation.

(*g*) Finally, a word concerning the ambiguities in the mathematical notion of function. Three definitions of the term *function* have been indicated in the preceding paragraphs: the extensional definition, the intensional, and a third derived from the second of these two. This number could be easily multiplied by taking account of the ambiguity of the term *symbol* used in the intensional definition. The ambiguity can be best illustrated by an example: When speaking of the symbol *the*, there is one sense (1) in which it is sensible to speak of five *the*'s occurring on one page as distinct symbols; there is another sense (2) in which there is just one symbol, *the*, in the English language, while the French language has three (*le*, *la*, *les*), and the German language six (*der*, *die*, *das*, *des*, *dem*, *den*); (3) there is a sense in which *le*, *la*, *les* are all instances of the same symbol; (4) there may be a sense in which *der*, *the*, *le*, are all instances of the same symbol. In addition it is possible to use the term symbol in such a sense (5) that a symbol is a particular sense datum. In this sense each time the inscription on a signpost is read and understood by any person it functions as a new symbol. These five differentiations by no means exhaust the possibilities of type token ambiguity[1] and are relevant to the analysis of the relation between the various notions of function.

[1] Cf. on this topic C. S. Peirce: "Prolegomena to an Apology for Pragmaticism," *Monist*, 1906, reproduced in part, with other relevant matter, in *The Meaning of Meaning*, Appendix D.

Propositional Functions

The intensional definition of function adapted to the propositional calculus and contrasted with Russell's definition.

The intensional definition of mathematical function will serve *mutatis mutandis* for the definition of functions in any system of symbols where variable symbols occur. Thus, in the propositional calculus, any symbol such as $p \vee q$ may be considered as a function of the symbols p and q which form part of it; an example of a function of one variable in this calculus would be the symbol $\sim p$. It is easily seen in this special case what symbols represent the notions already defined for the case of mathematical functions: the field of variation of p is the aggregate of all propositions, the *values* of the function will be the propositions which are the contradictories of the values of the independent variable p, p is the *argument* of the function, and finally the variable p occurs in both a formal and an illustrative use in the symbol $\sim p$. The sign $\sim$ is of course not a variable; it indicates the manner in which a typical proposition $\sim p$ is derived from a typical proposition p. Symbols like $p \vee q$, $\sim p$, etc., which occur in the propositional calculus, are called *truth functions* of their arguments because their truth or falsity depends only on the truth or falsehood, and not on the specific nature, of the values of their arguments.

Similarly, it will be possible to form functions whose values are propositions and whose arguments are variables capable of denoting *any* object; we thus arrive by analogy with the intensional mathematical definition of function at a definition parallel to, but not identical with, Russell's definition of propositional function. An example can be obtained by changing *the wall* in *the wall is red* into a variable x, furnishing a propositional function x *is red* of which the proposition *the wall is red* is one value. The function has two arguments, and could be represented by *x is y* where the *is* is the only portion of the symbol which is neither a variable nor a value of a variable and therefore shows the manner in which the function is constructed out of its arguments.[1] This, however, is not the view adopted by Russell, who considers, e.g. *x is red* to be a function of *one* argument only, analogous therefore to the function sin x. For convenience, let *is red* be represented by ϕ (ϕ being an illustrative variable); then ϕx means *x is red.* The function itself Russell represents by $\phi\hat{x}$ which may be read as 'ϕx blank'. He seems to have believed that the ϕ shows the form of a function of one argument in this case. There seems no good reason to assume, as Russell does, that the relational or predicative terms in a proposition must represent the form of the propositional function involved; this assumption serves not only to complicate the development of the calculus but leaves the whole notion of propositional function inconsistent and vague.[2]

[1] It should be noticed that *x is y* would not be an adequate generalization of *the wall is red* and would need to be supplemented by a statement restricting the fields of variation of x and y to objects of the same logical type as 'the wall' and 'red' respectively.

[2] Cf. W. E. Johnson, *Logic*, vol. ii, ch. 3, for this type of criticism of Russell's definition of propositional function.

The Calculus of Propositional Functions Resumed

Having defined propositional functions we can now proceed with our exposition of the functional calculus of *Principia Mathematica*; we shall give an account first of the new symbols introduced, then the theory of types and the axiom of infinity, and finally describe and criticize the axiom of reducibility. We begin with a number of definitions. These are based with occasional simplifications on those given in the introduction to *Principia Mathematica*. Their purpose is to facilitate and abbreviate the discussion of the theory of types and the axiom of reducibility to which the reader who is familiar with these definitions may therefore at once proceed.

Quantifiers, Truth-Values, etc.

Manipulation of the notions *some* and *all* in the calculus of propositional functions. This section and the four short sections which follow deal with the technicalities of manipulating the calculus of propositional calculus.

The notion of propositional function, which chiefly distinguishes the calculus of propositional functions from the calculus of propositions has already been described. In order to remain as close as may be to some standard logistic position in the following paragraphs, Russell's definition of propositional function will always be assumed: it will be remembered that a propositional function is so called because all its values are propositions. If all the arguments of a propositional function are replaced by definite values chosen from their respective fields of variation, the propositional function becomes a definite proposition which may be either true or false; if it is true, the propositional function in question is often said to have the *truth-value truth* for those values of its arguments, while if the substitution of those values of the argument produces a false proposition, the function is said to have the *truth-value falsehood* for those values of the argument. In other words the 'truth-value' of a true proposition is truth; of a false proposition, falsehood. The values of a function must not be confused with its truth-value, for the former are propositions while the latter is either truth or falsehood.

The calculus of propositional functions now introduces two new primitive ideas which roughly correspond to 'all' and 'there is a' and are necessary for the analysis of general and existential propositions. *All* and *there is a* are symbolized respectively by (x) and (Ex),[1] two symbols which are attached like indices to propositional functions, converting them into propositions, in a manner which two examples will make clear: If $L(x)$ means *a line passes through the point x*, $(x)L(x)$ means *all points have a line passing through them*, and $(Ex)L(x)$ means *there is a point through which a line passes*. Or again $(x)L(x)$ may be considered as equivalent to the simultaneous assertion of all the propositions $L(x)$, and $(Ex)L(x)$ as equivalent to the assertion that $L(x)$ has the truth-value truth for one at least of the values of its arguments. Since these new symbols are both primitive in the functional calculus it is not necessary to define them, provided the foregoing explanations have made clear how propositions whose expression in ordinary language would require the use of *all* or *there is a* are to be replaced by symbolic expressions containing (x) and (Ex) respectively. The two symbols thus introduced may for convenience be called quantifiers, and qualified by the

[1] In *Principia Mathematica* the E of (Ex) is written backwards; it has become quite usual, for the convenience of typography, to write the symbols as in the text above.

words general and existential respectively.[1] There are several points to be noticed about the use of quantifiers :—

(1) The quantifiers are for the present meant to apply only to individuals, i.e to objects which are values of arguments to propositional functions and are not themselves propositional functions, but not to propositional functions themselves. This use of the quantifiers is narrower than the use of *all* and *there is a* in ordinary speech, for the latter are often made to refer to predicates, as when we say "there is a certain colour I need", while predicates occur in the functional calculus of *Principia Mathematica* as propositional functions and cannot for the present be quantified. The calculus which results when this condition is satisfied will be called the *restricted functional calculus.*[1] Very soon it will also be necessary to consider the *general functional calculus.*[1]

(2) In order for propositions involving quantifiers to have an exact meaning the field of variation of the variable in question must be known and specified ; it is usually assumed in the functional calculus as in ordinary speech that the field of variation of the variables is the widest possible consistent with the condition that the corresponding values of the function are propositions and not nonsense, e.g. for the propositional function *x is an ocean, the North Sea* belongs to the field of variation of *x*, but *Wednesday* does not. In this way each propositional function determines fields of variation for its arguments[2]; if a narrower field than this is desired it is usually obtained by modifying the propositional function in question accordingly.

(3) A propositional function of several arguments will, by the successive application of existential or general quantifiers, give rise to several different symbolic constructs,

[1] These terms are not used in *Principia Mathematica.*
[2] This statement is subject to modification later owing to the theory of types.

which will, in general, represent a number of distinct propositions. In fact a proposition obtained by quantification of a function is not completely determined by specifying which of the various arguments of the function are to be made apparent by the application of existential and which by the application of general quantifiers, for the order in which these symbols are applied will in general be significant. For example let $F(x, y) = x$ is *the father of* y; then $(Ex)\ ((y)\ F(x, y))$ is the proposition *there is somebody who is everybody's father* but $(y)\ ((Ex)\ F(x, y))$ is the proposition *everybody has a father*. It is however easily seen that changing the order in a group of successive quantifiers, all of the same kind, does not alter the sense of the expression in which they occur. Such groups of quantifiers may therefore be written $(x, y, z, \ldots)$ or $(Ex, y, \ldots)$ respectively. Thus the expression

$$(x)\ (Ey)\ (Ez)\ (w)\ F\ (x, y, z, w).$$

where a number of brackets have been dispensed with in a manner which is sufficiently obvious, can also be written $(x)(Ey, z)(w)F(x, y, z, w)$ without ambiguity.[1]

(4) It may be noticed that any variable to which a quantifier has been applied in some context becomes an *apparent* one in that context.

(5) If a quantifier is placed before an expression containing several propositional functions it is necessary to indicate to which of these functions the quantifier is meant to apply; the *scope* of a quantifier is defined as the function to which the quantifier is meant to apply and is indicated in *Principia Mathematica* by dots bracketing the scope on to the quantifiers, e.g. in $(x) : \phi x . \supset . \psi x$ (or: ϕx *always implies* ψx) the scope of the quantifier is the function $\phi x . \supset . \psi x$, but in $(x) . \phi x : \supset . \psi x$ (or: *if* ϕx *is always true then* ψx *is true for the value* x) its scope is ϕx.

[1] These conventions, again, have come into general use since *Principia Mathematica*.

The Algebra of Propositional Functions

After the symbols mentioned in the previous paragraph have been introduced the algebra of propositional functions proceeds in the same fashion as the algebra of propositions. Starting with a number of tautologies, i.e. propositions involving propositional functions and true whatever propositional functions are substituted (just as the tautologies of the propositional calculus yielded true propositions for all values of p, q, etc.), we obtain new tautologies by the use of certain rules of manipulation. Apart from the additional complexity produced by the introduction of additional symbols for propositional functions, quantifiers, etc., the restricted functional calculus presents no features which have not already been discussed in the case of the propositional calculus; it may however be noticed that, whereas in the latter a uniform procedure has been found to determine which expressions are tautologies, so that manipulation of formulæ in that calculus may proceed without use of a definite system of axioms, this is not the case with the restricted functional calculus, where no uniform procedure is known for detecting tautologies. This makes the use of a system of axioms essential for the demonstration of tautologies in the restricted functional calculus.

In order to prepare the way for the definition of integers in terms of logical notions one or two further definitions are necessary. They include definitions of *extensional propositional functions* (not to be confused with the previous *extensional definition of mathematical functions* which has already been discussed), important for their bearing on the question of the necessity for an axiom of reducibility in the logistic systems, and definitions of *incomplete symbols*, the last of which include *classes* and *descriptions* as special cases.

Extensional Propositional Functions

When an implication, say $\phi(x) \supset \psi(x)$, holds between two propositional functions for all values of the argument x it is said that $\phi(x)$ *formally implies* $\psi(x)$. Two propositional functions are said to be *equivalent* if each formally implies the other; in terms of the symbols already defined this may be expressed as

$$(\phi(x) \equiv \psi(x)) = (x)\{(\phi(x) \supset \psi(x)) . (\psi(x) \supset \phi(x))\} \quad Df.$$

If some or all of the symbols for propositions in a truth function of propositions be replaced by symbols for undetermined values of propositional functions, *truth-functions of propositional functions* are obtained, e.g. $\phi(x) . \psi(x)$ will be a truth-function of ϕ and ψ. In general, a truth function of propositional functions is defined as a symbol which contains the propositional functions and whose truth-value depends only on the truth values of these propositional functions.

Propositional Functions of Functions and the General Functional Calculus

By analogy with the representation of properties of and relations between individuals as propositional functions, properties of and relations between propositional functions can in turn be represented by functions whose arguments are variable propositional functions. From previous discussion on the nature of functions in general, it will be clear that any expression in which symbols for a propositional function of individuals occur as illustrative variables may be regarded as a function of propositional functions; if the expression in question is such that when the variable propositional functions occurring in it are replaced by specific functions the resulting expression is a proposition, we shall have a *propositional* function of functions. Hence any variables denoting *individuals* in such an expression must be apparent. Examples of such propositional functions of functions would be the expressions $(x)(\phi x . \psi x : \supset . \phi x)$ or ϕa where a is the name of some individual. The introduction of such propositional functions of functions and, where occasion arises, of quantifiers attached to them produces a calculus of propositional functions wider than the restricted calculus already described.

Extensional Functions of Functions

A propositional function of functions is said to be *extensional* if it has the same truth-value for all arguments which are formally equivalent.

At first sight it may appear obvious that some functions of functions are not extensional as here defined. Consider, for example, the propositional function ϕx *has seven letters* one of whose values is the proposition '$\hat{x}$ is a man' *has seven letters*. If in this proposition $\hat{x}$ *is a man* be replaced by the formally equivalent function $\hat{x}$ *is a featherless biped* the truth-value of the proposition changes from truth to falsehood. So an example of non-extensional function appears to have been constructed. Wittgenstein (*Tractatus Logico-Philosophicus*) and Carnap (cf. *Der Logischer Aufbau der Welt*, p. 62), among others, have asserted that all functions of functions are extensional. Cf. p. 122.

Derivation of Mathematical Functions from Propositional Descriptions

In the logistic thesis the problem of adequately symbolizing mathematical expressions reduces in general to the problem of analyzing the inter-connections between propositional functions and descriptions. The logistic solution involves difficulties associated with the definition of identity.

It is now our business to consider how mathematical functions are derived from propositional functions in the logistic system of *Principia Mathematica* ; it will be a simplified account, reserving for subsequent discussion complications produced by the *Principia* definition of identity ; for the present no difficulty will be caused by treating identity as a primitive or fundamental notion.

We need to consider two principal types of expression involving mathematical functions : (1) expressions of which sin x is a typical example, where the variable is real and occurs in its illustrative use, and (2) expressions such as sin $(\pi/2)$, derived from expressions of type (1) by substituting a constant for the variable. The *Principia* view is to regard sin $(\pi/2)$ as a definite description of the number 1, since sin $(\pi/2) = 1$; sin $(\pi/2)$ is asserted to bear the same relation to the number 1 as *the present King of England* to King George V.

Further, the distinction made between expressions such as sin x of type (1) above and expressions such as sin $(\pi/2)$, is to regard the latter as completely determinate or *definite* descriptions of some unique number, and the former as *indefinite* descriptions of some unspecified number comprised in the field of variation of x.

Since mathematical functions and their values are thus considered to be analogous to propositional functions in general, the problem of reducing mathematical functions to propositional functions becomes a special case of the very general problem of exhibiting the connection between propositional functions and descriptive phrases and explaining how the latter may be derived from the former.

In order to render as simple as possible the account of the manner in which this is accomplished in *Principia Mathematica* it is best to start with some specific descriptive phrase, say *The present King of England.* Instead of analyzing this phrase in isolation a rule is given for symbolizing any proposition in which *the present King of England* occurs. Consider the proposition *The present King of England lives in Buckingham Palace* for example; this is analyzed into the conjunction of the two propositions *There is one and only one x such that x is the present King of England and x lives in Buckingham Palace. x lives in Buckingham Palace* and *x is the present King of England* are propositional functions of one variable of the form ϕx and the statement *there is one and only one x satisfying* ϕx is symbolized by

$$(Ex)(y)(\phi y \equiv (y = x) . \phi x)$$ [1]

It is important to notice (*a*) that the *is* which occurs in "there is one and only one *x* such that . . ." and is symbolized by (Ex) has a different meaning from the *is* which occurs in "*x* is the present king of England"; for the first denotes the existence of a particular while the second denotes what W. E. Johnson refers to as the characterizing tie, viz. the characteristic and indefinable manner in which a particular is attached to a quality which qualifies it.

(*b*) Identity is treated as a propositional function of two arguments. There is clearly some difficulty here since to say two things are identical is merely a clumsy way of asserting

[1] i.e. "Some *x* satisfies ϕ and all things which satisfy ϕ are identical with that *x*."

that there is in reality only one thing. It is difficult to see how identity is to be regarded as a relation between two things, or, if it is not, what then becomes of the logistic definition of descriptive phrases.[1] This is to some extent overcome in *Principia Mathematica* by defining two things as identical if they have all their properties in common. Two objections arise immediately: first that the definition is incorrect since even if it is never true that two distinct things have all their properties in common it is yet significant to assert that they have.[2] And, secondly, on account of the contradictions which the theory of types (p. 101) was invented to eliminate, it is not permissible in the logistic scheme to speak of *all* the properties which two things have in common. The second objection is met in *Principia Mathematica* by the use of the axiom of reducibility which considerably complicates the final definition.

Plural Descriptive Phrases

Plural descriptive phrases are derived from propositional functions by the technical device of definitions in use involving the use of 'incomplete symbols'. The limitations of this method are noted.

Descriptive phrases of the type so far discussed, viz. *the so-and-so*, can be derived only from propositional functions satisfied by one and only one argument; and, conversely, every such propositional function gives rise to a descriptive phrase of this kind. It is, however, easy to apply similar considerations to propositional functions satisfied by more than one argument, and thus to obtain plural descriptive phrases analogous to the descriptive phrases already

[1] It has been proposed to do altogether without the use of 'identity' under discussion. Thus, e.g. Wittgenstein says "5.53, Identity of the object I express by identity of the sign and not by means of a sign of identity. Difference of objects by difference of the signs" *Tractatus Logico-Philosophicus.*

[2] L. Wittgenstein, op. cit., 5.5302.

mentioned; the plural descriptive phrase denotes all the arguments satisfying the propositional function just as a uniquely descriptive phrase of the form *the so-and-so* denotes the single argument. That is to say a plural descriptive phrase denotes what would usually be called 'the class of all the arguments' satisfying the propositional function considered; classes therefore enter the logistic scheme through plural descriptive phrases.

Speculations concerning the nature of classes and the associated problem of the connection between the extension and intension of classes, to use the traditional terminology, have presented great difficulties to logicians and have received as yet no adequate resolution.[1] The difficulties involved in answering such questions can however be avoided by transforming the symbols called propositional functions in the spirit of the technique of formal analysis which we have already explained.

Choice of the transformation appropriate for the expression of classes is facilitated by the fact that the distinction between a predicate and the objects it qualifies is not a discovery of logicians but is already made in the unsophisticated language of common sense. This is shown by the possibility of converting such a statement as *red is a colour* into *all red things are coloured* (*things*). For the purpose of reducing mathematics to logic it is sufficient to invent a self-consistent symbolic mechanism for exhibiting this distinction systematically and quite unnecessary to speculate upon the ontological significance of this distinction.

In the case of *the x which satisfies* ϕx the symbolism chosen

[1] "Extension, as used in relation to intension, is an extremely ambiguous word. The traditional treatment of this topic is very unclear owing to the fact that quite different notions have been confused, and the topics connected with each of them have been dealt with together. These confusions run throughout the traditional logic which is based upon the metaphysical theories implicit in Aristotle's theory of logic." L. S. Stebbing, *A Modern Introduction to Logic*, p. 28.

is $(\imath x)\ (\phi x)$; the transformation consists in the first place of adding the same pseudo-quantifier $(\imath x)$ to any propositional function ϕx. This apparently trivial alteration modifies the form of some of the propositions in which ϕx can occur and leads to a considerable simplification of theorems. In the same manner we represent *the x's which satisfy* ϕx by $\hat{x}\,(\phi x)$.

New symbols cannot be derived from old in this fashion quite arbitrarily. When defining symbols it is necessary first to indicate which features of such symbols are significant, i.e. to state in which circumstances two such symbols are regarded as identical and, secondly, to indicate the contexts in which the symbols may be correctly employed.

In the case of classes the answers of *Principia Mathematica* to these two demands are :—

(1) Two classes are said to be identical if the propositional functions from which they are derived are equivalent in the technical sense of equivalence previously defined (p. 44); and

(2) Though no explicit statement concerning the contexts of classes is made, the most important contexts are in fact (*a*) of type $a \epsilon \hat{x}\,(\phi x)$ which means *a is a member of the x's satisfying* ϕx, *a* being of the same type as the arguments to ϕ, and (*b*) $\hat{x}\,(\phi x) . \hat{x}\,(\psi x)$ which means *the x's satisfying the function* ϕ *and the function* ψ *simultaneously*.

The statement of the significant features and possible contexts of a newly-defined symbol means that the choice of such symbols is subject to limitations which must be investigated before the symbol can be safely employed. For it may be that the definition of the symbol is inconsistent either with the rules of identity, as defined in (1) above, or with the rule stating the contexts to which it is restricted. These conditions are not discussed in *Principia Mathematica*.

The reason for this omission is insufficient recognition of the distinction between formal and non-formal analysis upon which we have already had occasion to remark. Before long we shall return to the question of the consistency of the *Principia* definition of classes.

Definitions of Descriptions and Classes

The definitions are of the kind termed ' definitions in use ', that is, a definition is given, not of the symbol to be defined, but of certain expressions containing it. Though it is not possible to replace the symbol itself by symbols already defined, a rule is given for translating every expression in which it occurs into expressions containing only symbols previously defined.

We have seen that *the x satisfying* ϕx is symbolized by $(\imath x)(\phi x)$ and *the x's satisfying* ϕx by $\hat{x}(\phi x)$; it is also necessary to indicate the *scope* of these expressions, i.e. the proposition to which $(\imath x)(\phi x)$ is to be considered as belonging. This is achieved by prefixing the pseudo-quantifier $[(\imath x)(\phi x)]$ to such expressions, with sufficient dots to bracket the scope.

For the sake of economizing symbols the convention is made that the pseudo-quantifier $[(\imath x)(\phi x)]$ may be omitted when the scope of $(\imath x)(\phi x)$ is the smallest propositional function containing it. (*Principia Mathematica*, i, p. 181.)

Omitting the complications due to the *Principia* definition of identity (involving the use of the axiom of reducibility) the definition of a proposition containing $(\imath x)(\phi x)$, say $\psi\{(\imath x)(\phi x)\}$, becomes

$$\{[(\imath x)(\phi x)]\,\psi\,(\imath x)(\phi x)]\} = \{(Ey)(x)(\phi x \equiv x=y)\,.\,\psi y\,.\,\phi x\}\ Df.$$[1]

i.e. any statement ψ about the x which satisfies ϕx means: one and only one thing does as a matter of fact satisfy ϕ and ψ is true of that thing.

It will be noted that the definition is so chosen that if ϕx is not satisfied by exactly one argument any proposition

[1] This is *Principia* definition 14.01 simplified.

containing "the x which satisfies ϕx" is false (and not meaningless as it should strictly be).

Similar symbolism is adopted for classes. $\hat{x}(\phi x)$ means *the x's which satisfy* ϕx and any phrase in which it occurs, say $\psi\{\hat{x}(\psi x)\}$ is defined by

$$[\hat{x}(\psi x)]\, \psi\{\hat{x}(\phi x)\} = \psi\{\phi\hat{x}\}.$$

Complete and Incomplete Symbols

This and the next three sections continue the discussion of incomplete symbols. Russell's definition is stated and rejected in favour of a more precise definition.

The occurrence in *Principia Mathematica* of 'definitions in use' leads to a distinction between complete and incomplete symbols. According to the definition given there, the latter are such as have no meaning in isolation and cannot be legitimately used without the addition of further symbols. Examples of incomplete symbols in this sense would be the mathematical symbols for multiplication and addition, $\times$ and $+$, which are used only in contexts such as $2 + 4$, $a \times b$, etc.

This definition is however unsatisfactory for the following two reasons:—

(1) It follows from the definition that any propositional function symbol, say f, is incomplete, since it requires the addition of one or more arguments $x, y, \ldots$ to complete its meaning by an indication of the number of variables on which it depends, and propositional functions were not intended to be incomplete symbols.

(2) To say incomplete symbols are such as have no meaning in isolation is insufficiently precise language for a definition. In one sense no symbol can occur in isolation, for it must be capable of combination with other symbols of the system to which it belongs. It will be accompanied by such symbols in all contexts in which it occurs; its syntax is part of its 'meaning'.

The definition of incomplete symbols can be improved either by replacing 'meaning' by some more precise notion or by eliminating the reference to meaning altogether in order to provide a formal definition based on the manner in which symbols can occur in systems. The first method has been adopted by Professor G. E. Moore [1]; the second will be chosen here.

Definition of Incomplete Symbol

The road from logic to mathematics runs from propositional functions to descriptions to classes to integers; classes have suffered vicissitudes and the change from collections existing in their own right to incomplete symbols manufactured from symbols needs safeguards.

A symbol is complete in a given system of symbols if it is either undefined, i.e. occurs in the axioms of the system, or else is defined in such a manner that it can be replaced in every context in which it occurs by a group of defined symbols. In a definition of a complete symbol, the *definiens* is a group of symbols specified independently of any context.

An *incomplete* symbol is one whose definition consists of a rule for transforming any expression in which it occurs into an expression containing only complete symbols, the manner in which this transformation is effected depending on the context of the incomplete symbol.

These definitions agree with the usage of *Principia Mathematica* and all symbols 'defined in use' will be incomplete. From this point of view the use of classes and

[1] Professor Moore defines *incomplete in a certain usage*; the definition is unpublished but is quoted by Miss Stebbing in *A Modern Introduction to Logic* (p. 158) as follows: "*S*, in *this* usage is an incomplete symbol" = "*S*, in *this* usage, does occur in expressions which express propositions, and, in the case of *every* such expression, *S* never stands for any constituent of the proposition expressed." This definition involves the notion of *constituent of a proposition* which needs further explanation.

descriptions in a logistic system is purely one of convenience. Nevertheless, the use of incomplete symbols is important for the following reasons :—

Importance of Incomplete Symbols

(1) In spite of the fact that incomplete symbols can be replaced by complete symbols in every context and are therefore theoretically unnecessary their introduction enormously abbreviates complicated demonstrations. The amount of paper occupied by the first part of the *Principia Mathematica* would no doubt be of astronomical dimensions if the use of incomplete symbols were forbidden. This favourable characteristic is shared by all symbols whether complete or incomplete which are defined in order to be used in demonstrations.

(2) More important than this saving in space and the consequent facility of manipulation is the fact that transforming complicated theorems composed of complete symbols into comparatively simple theorems containing both kinds of symbols leads to the discovery of formal analogies between incomplete and complete symbols. New, incomplete symbols are found to combine in modes identical with the laws of combination of symbols previously studied ; once such a correspondence has been established, sets of theorems already proved can be transformed at one stroke into theorems concerning the new symbols. The advantages of this technique are clear ; it provokes the discovery of unsuspected relationships and a profounder comprehension of the interdependence of diverse fields. The calculus of classes offers striking examples of such analogies in the formal similarity of the operations of class-addition and class-multiplication to the operations which bear the corresponding names in the propositional calculus.

Ontological Status of Incomplete Symbols

A few remarks may be added concerning the 'reality' of classes and other objects denoted by incomplete symbols.

Though it is a fact which belongs to psychology rather than to logic it is noteworthy that since incomplete symbols appear to behave like complete symbols and eventually appear in expressions which contain only incomplete symbols, the latter attain the status of complete symbols in the opinion of those who manipulate them, i.e. they are considered to denote 'real things'.

The metaphysical respectability of the things which incomplete symbols denote, though it appears to need the successful incorporation of the symbols into a calculus, is not guaranteed when this demand is satisfied and appears to depend on subjective factors which include the following :—

(1) The extent to which the symbol in question is used and finds applications: the greater the number of applications to and analogies with other symbols already accepted as denoting real entities, the more pretensions to reality our incomplete symbol acquires.

(2) The decision whether the introduction of incomplete symbols in any given case leads to the discovery of genuine mathematical entities or is merely a technical trick with no further significance is influenced by the possibility of remodelling whole systems, containing both complete and incomplete symbols by a new choice of axioms into a new system in which some of the previously incomplete symbols now appear as complete. If the new system is valid such incomplete symbols will gain in respectability.

Nature of *Principia* Classes

Turning now to consider the nature of classes in *Principia Mathematica* in order to decide whether the introduction of incomplete symbols in this connection is a valid device, we find a curious position. For classes were introduced in Russell's earlier expositions of the logistic thesis as aggregates or collections of objects. This fact was chiefly responsible for the paradoxical flavour of early logistic definitions of integers; for an integer is a class of classes and hence, originally, a collection of collections of objects. Yet, for compensation, no need was felt to prove the reality of such classes; that was regarded as self-evident. Such a theory made the truth of mathematics contingent upon the existence of sufficient objects in the universe of perception and required a special axiom of infinity to that effect. And what was meant by the existence of a class remained unanalyzed and unanalyzable. This theory collapsed through internal inconsistencies associated with the existence of infinite classes, and was succeeded by many alternative theories of classes all less realist than that described above, until classes eventually came to be degraded to incomplete symbols.[1] But no attempt was then made either to give a fresh discussion of the ontological status of classes or, alternatively, to verify that the definition was technically free from defects.

Consistency of Definition of Classes as Incomplete Symbols

To interpret classes as incomplete symbols is a *tour de force*, needing to be safeguarded against inconsistency by methods here discussed.

When classes are regarded in such a light that their introduction is purely a technical device, emphasis shifts to

[1] Russell: "It is reasonable to regard the theory . . . as right in its main lines, i.e. in its reduction of propositions nominally about classes to propositions about their defining functions. The avoidance of classes as entities by this method must, it would seem, be sound in principle" (*Intro. to Mathematical Philosophy*, 1919, p. 183). But ten years previously in the *Principles of Mathematics* he was saying "When a class is regarded as defined by the enumeration of its terms it is more naturally called a collection" (p. 69) and "a class we agreed is essentially to be interpreted in extension; it is either a single term, or that kind of combination of terms which is indicated when terms are connected by the word *and*" (p. 80).

the validity and self-consistency of this technique. It is not permissible to manufacture incomplete symbols arbitrarily. If by mischance the definition of incomplete symbols were to lead to results inconsistent with the axioms of the propositional calculus all theorems in whose demonstrations classes occurred would be suspect, and classes would become of no value as a symbolic device. This question, then, though completely ignored in *Principia Mathematica*, is of extreme importance in the rigidly deductive scheme which the logistic definition of mathematical notions aspires to be. It is necessary to be quite explicit on this point, even at the risk of wearying the reader, for it is a major issue in deciding upon the virtues of *Principia Mathematica*. It will be recalled that the definition of the class associated with some propositional function, ϕx say, in the simplified form adopted for the present discussion is

$$[\hat{x}\,(\phi x)]\;\psi\,\{\hat{x}\,(\phi x)\} = \psi\,\{\phi x\}.$$

The ϕ and ψ which occur in this formula play very different parts; ϕ is merely a propositional function proper, one of the objects contained in the subject-matter of the axioms, and need have no further meaning than that, when the *correctness* of the deductions of *Principia* is investigated. ψ on the other hand is a shorthand symbol to replace a set of words; it means *any expression in which* $\hat{x}(\phi x)$ *occurs* and in applying the formula above it is essential that the meaning ψ should be so understood.

The validity of introducing classes appears at least dubious when viewed in this light; for if it were true, as it is not, that every expression containing class symbols can be transformed into one not containing such symbols, such a fact could be perceived only by 'intuitive induction', i.e. by direct recognition of the validity of such a transformation in all the infinitely varied cases which might arise.

The number of types of possible expressions containing class symbols is infinite so that the validity of any technical device applicable to *all* such expressions must be based either on (1) direct recognition that *every* such expression is capable of being transformed in the manner required, or on (2) a proof bv ' mathematical induction ', i.e. one which proceeds as follows :—

(*a*) The definition is verified to be consistent for some set of simple expressions from which all other expressions can be built up by the use of certain principles of construction (e.g. application of quantifiers, increase in the number of variables, etc.), and (*b*) it is proved that the growth of an expression by the application of any such principle leads to no inconsistencies. If (*a*) and (*b*) can be demonstrated, the definitions can be seen to be consistent for *any* given expression on applying the proofs referred to under (1) and (2) a finite number of times. Method (1) which we have referred to above by the name ' intuitive induction ' is specially applicable to unorganized collections, method (2) to organized infinite sets of expressions.

In regard to the definition of classes in *Principia Mathematica* the situation is as follows: method (1) cannot be applied; for the expressions which can be constructed from the materials of the calculus of propositional functions are too complex to permit of any such general survey as that method requires. Further, without additional restrictions on the possibility of constructing expressions containing classes the definitions are inconsistent and lead to contradictions.[1]

[1] It may perhaps be objected that in the actual demonstrations which occur in *Principia Mathematica* the number of expressions containing class symbols must be finite, and that it is unnecessary to establish the correctness of the definition for all such expressions if it can be seen to be valid in the case of the finite number which actually occur. Our answer must be that unless the definition is restricted to apply to the expressions which actually occur in those specific proofs the definition must be a consistent one for all expressions which can be constructed. Otherwise, a contradiction could be demonstrated inside the logistic calculus and eventually in mathematics; there is good cause to assert that the unrestricted use of incomplete symbols does produce such contradictions.

Russell and Whitehead invented the Theory of Types to eliminate the contradictions due to the unrestricted use of classes; their remedy is based on a theory of the illegitimacy of the notion of 'all' and will be discussed later. The effect of the theory of types and its associated symbolism is to impose additional order upon the unorganized assemblage of expressions in which classes may occur, and to prevent some possibilities of inconsistency by forbidding certain types of expressions. Nevertheless, it is still not possible to enumerate the principles of construction which are needed for method (2) above, and even after the introduction of the theory of types there remains no guarantee that the conventions are consistent. In fact the authors of *Principia Mathematica* seem nowhere to have recognized that any purely symbolic device such as the introduction of incomplete symbols, or even the omission of the pseudo-quantifiers which precede them and indicate their scope, needs justification; the impossibility or extreme difficulty of establishing the validity of the *Principia Mathematica* definitions of incomplete symbols is due to the vagueness of the notion of propositional function already discussed.

The pertinence of these objections is very strikingly shown by Dr. Chwistek's discovery that the apparently innocent convention for omitting scope indicators is inconsistent and has to be abandoned.

Attempts have been made to remedy these defects of *Principia Mathematics* in at least two ways: (*a*) by restricting the logical calculus of propositional functions to the so-called 'restricted' calculus already discussed, and demonstrating the consistency of all conventions used (Hilbert); (*b*) by giving a constructive definition of propositional functions to permit of the application of mathematical induction. Of these the first involves the rejection of the reduction of mathematics to logic.

Discussion of the use of incomplete symbols in the logistic scheme might usefully be supplemented by consideration of the analogous occurrence of *abstraction* and *ideal elements* in mathematics which in turn assists in understanding how mathematical objects are derived from logical.[1]

[1] An excellent account is to be found in Professor H. Weyl's *Philosophie der Mathematik* (§ 2: "Die aufbauende Mathematische Definition").

The Real Number

At a critical point in the logistic system surprising contradictions appear and must be expelled ; the problems involved are associated with the mathematical theory of continuity, based upon 'intuitions' whose exact nature is always conveniently vague.

This section will exhibit the connection between the so-called mathematical paradoxes and the logistic construction of real number. In the course of the account it will be maintained that the contradictions which occur in the logistic scheme cannot be regarded as analogous to 'slips' in mathematical proofs, possibly to be eliminated by increased care with definitions and substitutions. They are not fortuitous blemishes but difficulties inherent in the conception of an actual or extended infinity, a notion whose uncritical assimilation into the logistic scheme reproduces in a new form the very difficulties which are already familiar to the mathematician. The treatment of *Principia Mathematica* and the logistic philosophers in general has clarified the questions which are involved but has not succeeded in eliminating the difficulties.

We begin by considering the relation between the notions of *real number* and the *continuum*. The real number, is a concept intimately connected with that of the continuum and enters into that part of pure mathematics which is specifically concerned with problems arising from the analysis of continuity. Of this domain the most important for present purposes consists of the infinitesimal calculus and the modern theory of functions which are usually grouped together as analysis, a term which excludes both arithmetic and geometry. The relation between these three disciplines can be expressed summarily by stating that the method of analysis essentially consists of applying arithmetical methods to the manipulation of certain geometrical intuitions of continuity.

In order to amplify this statement, whose conciseness would otherwise be misleading, it is necessary to supply a more detailed description of the so-called intuitions of continuity. It is a remarkable fact that, although the mathematical theory of continuity is alleged to be based on direct experiences of continuity, no descriptions of these alleged experiences are to be found in the literature of mathematical philosophy. The following is an attempt to describe some at least of the features of these experiences. We shall not need to make any assumptions at this stage concerning the psychological or epistemological status of these 'intuitions'. It will be convenient to confine our attention to the visual field where intuitions of continuity are least vague, not prejudging, however, questions of the existence and status of intuitions of continuity in the fields of sensations associated with sense organs other than the eye.

'Intuitive' in the sense used here is to be translated approximately by 'direct' and 'not arrived at by a process of reasoning'. The purpose of our inquiry does not demand a more exact description of the meaning of this term.[1]

[1] A distinction needs to be made between direct or intuitive experience of continuity of the visual field, i.e. of a field of sense data, and between intuitive (i.e. not based on premisses) beliefs as to the continuity of physical space. These two senses of continuity are often confused, e.g. by Weyl who uses 'continuum' sometimes for portions of physical space (or space time) as in the sentence "Davon zu unterscheiden ist seine Verwirklichung an einem konkret vorliegenden Kontinuum, *wie es die räumliche Strecke ist* [*Philosophie der Math.*, p. 43—my italics] and sometimes for a continuum of sense data, e.g. in supporting Brouwers objections to the *tertium non datur* "Das passt sehr gut zu dem Charakter des anschaulichen Kontinuums; denn in ihen geht das Getrennt-sein zweier Stellen, beim Zusammenrücken sozusagen graduell, in vagen Abstufüngen, über in die Ununterscheidbarkeit" (ibid.). Presumably in stating "Die Mathematik gewinnt mit Brouwer die höchste intuitive Klarheit" it must be to the (alleged) connection between Brouwer's analysis of continuity and direct intuitions of continuity to which he is referring; but he nowhere supplies a precise description of these intuitions.

Intuitions of Continuity in a Sensory Field

'Intuitions' of continuity are here analyzed into an apprehension of connectivity and the possibility of indefinitely continued division; neither is directly observed and both must be translated, in any accurate logistic analysis, into statements concerning the multiplicity of symbols denoting portions of the continuum.

First it may be asked whether any such intuitions exist. Before the elaboration of logistic ideas geometry and, through geometry, analysis, was universally believed to be based on some geometrical intuitions such as we are trying to discover. Thus Dedekind in formulating his mathematical analysis of continuity said: "Es ist mir sehr lieb, wenn Jedermann das obige Princip [i.e. the principle of continuity we shall soon have occasion to describe] so einleuchtend findet und so übereinstimmend *mit seinen Vorstellungen von einer Linie*; denn ich bin ausser Stande irgend einen Beweis für seine Richtigkeit zu bringen, und Niemand ist dazu im Stande" (*Stetigkeit und Irrationale Zahlen*, p. 11). That is to say the principle of continuity is obviously true (einleuchtend) because it agrees with everybody's representation or conception (Vorstellung) of a line.

Although the continuum which is the subject of mathematical inquiry is in the first place a geometrical continuum (an ideal continuous line composed of points) and, eventually, an arithmetical continuum, i.e. a collection of real numbers, it is necessary to begin with intuitions of continuity in the visual field.

Intuition of the continuity of the visual field consists in apprehending (*a*) the connectivity of various portions of the field and (*b*) the possibility of infinitely dividing any portion of it. The field is conceived to have no gaps, to hang together, and to be capable of division into successively smaller portions.

On examining in more detail what is meant by (*a*) and (*b*) we observe that in one sense of 'connected' the visual field is certainly disconnected and not free from gaps; this is the sense in which the blind spot would be said to constitute a gap in the field. But this is not the sense in which the intuition that the visual field is connected formed the starting point for the formulation of mathematical continuity; it is connected in the sense that "we can't *see* any gaps in it". And in *this* sense the statement that visual space is continuous is a tautology. There is, however, another genuine sense in which we have intuitions both of visual connectivity and of visual disconnectivity, not of the parts which compose the whole visual field but of elements which constitute a selection or abstraction from it; e.g. it may be observed that one band of coloured light consists of strips of various shades of red connected without the intervention of other colours while another consists of strips of red separated by strips of blue; the red in the first band would be said to be connected and the red in the second band would be said to be disconnected. And the so-called connectivity of the visual field is derived from the intuition of the manner in which patches of the same colour (possibly of different shades) may be either separated or in proximity. Hence part of an analysis of what is meant by saying that the visual field is connected might include the statement: given any portion of the visual field there is another portion of the field bearing to the first portion the relation of contiguity, i.e. the relation between red patches of various shades when no other colour separates them. Hence the important result that the connectivity of visual space is in no way a property of the field taken by itself, but a relation between the field and 'portions' of it. Whether the field is connected or not depends on what is meant by a portion of it; if the portions were differently defined the visual field might become disconnected.

As for the alleged intuition of the infinite divisibility of visual space, it may be doubted whether any such knowledge is furnished by intuition. The possibility of endless division appears to be a schematic programme abstracted from the (directly apprehended) relation of one portion of visual space to another which contains it. When it is postulated that *any* portion of visual space could contain a smaller portion, it is difficult to understand how this fact, if it were a fact, could be apprehended directly. Here, as in the case of connectivity, the 'intuition' consists of postulating certain hypothetical relations between portions of the visual field, these hypothetical relations being based upon the relation of 'containing' or 'including' actually observed between some portions of the field.

Our conclusion is therefore that both the 'connectivity' and 'infinite divisibility' of the visual field are forms of the various ways into which the field can be regarded as divided into portions, i.e. if for any such division not only the portions but the relations of contiguity and 'containing' between them were symbolized, the 'connectivity' and infinite divisibility could be translated into statements concerning the multiplicity (cf. defn. p. 33) of all sentences constructed from these symbols.

Continuity in Geometrical Space

The evolution of the notion of continuity from Greek mathematics through Dedekind to *Principia Mathematica*. Uncritical application of the notions of connectivity and infinite divisibility to the space of geometry, conceived as real, provokes paradoxes (Zeno). The connection between continuity and the possibility of measurement leads to the discovery of a mathematical device for comparing incommensurable numbers (Eudoxus) and so eventually to a purely arithmetical conception of the continuum. The Achilles-Tortoise paradox and Eudoxus' construction are discussed.

The next stage in the formulation of mathematical continuity is the transition from the continuity of the visual field to the continuity of space. At the level of Greek mathematics the connection between the two was inevitable, for the geometry of Euclid was never doubted to be the geometry of actual space, i.e. an idealization by subtraction of irrelevant details, optical illusions, etc., of the geometry of the sensory fields. Yet the application to physical space of the concepts of connectivity [1] and of infinite divisibility immediately produced contradictions. For on the one hand it seemed necessary to demand the infinite divisibility, if not of matter, then certainly of space. If, however, space was real, it could in no way be regarded as something unfinished or in the process of becoming and the application of the concept of infinite divisibility to it would seem to be unjustified. The application of the notion of infinite divisibility to reality conflicts acutely with the recognition that this divisibility is essentially a process.

In their most acute form these difficulties were formulated in Zeno's paradoxes: Achilles can never catch the Tortoise if he starts behind it; for when Achilles has reached the position where the Tortoise started, it has advanced a little; and when he has reached that second position, it has moved a little farther forward. Thus Achilles, in order to pass the Tortoise, must actually perform an infinite number of acts, which is impossible.

This demonstration very clearly exhibits the contradictions produced by the notion of the reality of the extended infinite. It may be expressed in another form: if a line in space actually consists of infinitely many points, no motion at all is possible, for the smallest shift of position would involve the crossing of infinitely many points, i.e. the actual performance of an infinite number of acts.[2]

[1] The difficulties produced by the application of connectivity to physical space were principally centred around the possibility of empty space and are of less importance for the present discussion.

[2] Russell's discussion of the Achilles-Tortoise paradox (*Principles of Mathematics*, p. 350) takes an alternative and perhaps less interesting interpretation. "The slower" he says "will never be overtaken by

Zeno's paradoxes attacked not only infinite divisibility (i.e. the existence of the infinitesimal) but also finite divisibility (i.e. atomicity). Nevertheless, in the conflict between atomicity and infinite divisibility the choice made both in the geometry of Euclid and in the development of pure analysis in the nineteenth century was that of accepting the actual infinitesimal. Thus the geometrical line was regarded as composed of infinitely many points, the end-products of the infinite dividing process, conceived *per impossibile* as actually completed.

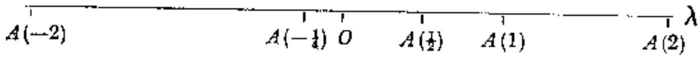
The importance of the notion of indefinitely continued divisibility of continuous lines in geometry is due to its connection with the theory of measurement, i.e. the possibility of the exact specification of congruent stretches [1] of lines by means of numbers, which we now proceed to explain.

With the help of the notion of congruence it is in the first place easy to explain how to obtain from any given stretch L another stretch L' whose length is any integral multiple of the length L. It follows immediately that the lengths of two stretches L_1, L_2, can be compared if each is an integral multiple of the length of L; for if $m \times L_1$ is congruent to $n \times L_2$, m and n being whole numbers, and $m \times L_1$ denoting a stretch whose length is m times that of L_1, the length of L_2 will be m/n of the length of L_1. In particular, if some arbitrary stretch L_0 be taken as a unit of reference, all stretches L such that two numbers m, n, can be found to specify their length in the manner described (i.e. all stretches commensurable with L_0) can be assigned a fraction m/n to specify their length. If on a line λ some point O is taken

the swifter, for the pursuer must first reach the point when the fugitive is departed, so that the slower must always necessarily remain ahead" (ibid.). The comment which immediately follows is "When this argument is translated into arithmetical language, it is seen to be concerned with the one-one correlation of infinite classes" (ibid.). Thus Russell in his discussion accepts precisely the attitude of the extended infinite which the paradox, in our interpretation of it, attacks."

[1] A stretch means a finite portion of a line between two points.

as origin, points $A(\frac{1}{2})$, $A(2)$, $A(\frac{3}{4})$, . . . can be associated in this way with each positive and negative fraction.



The Greeks discovered the existence of incommensurable lengths, e.g. the fact that in an isosceles right-angled triangle whose sides are equal in length to the unit stretch, L_0, the hypotenuse is incommensurable with L_0. Nevertheless, geometrical intuition shows that the hypotenuse in question can be transferred to the line λ, i.e. that there is some point B in λ such that the stretch OB is congruent to the hypotenuse.

The incommensurable lines appeared, in fact, to have the same status as any other lines for the following reasons :—

(1) They could be geometrically constructed.

(2) The incommensurable lines could be approximated to as closely as desired by commensurable lines ; i.e. in the representation above, commensurable points A_1, A_2, to the left of the point B, and commensurable points A_1', A_2', to the right of B, can be obtained, and such that the distance A_nA_n' becomes as small as we please for sufficiently large n.

The part which (1) plays in the so-called geometrical intuition of continuity is often forgotten ; it is, however, a mistake to imagine that (2) alone will furnish the existence of incommensurable lengths. The mere existence of the two converging series of points $A_1, A_2, \ldots A_n$, and $A_1' A_2', \ldots A_n'$ furnishes no intuitive evidence of the existence of a point to fill the gap. Intuition cannot discover the existence of the (infinitely many) gaps left in the line λ even after all the points obtainable by constructions in Euclid's geometry have been named.[1]

[1] Cf. also Galileo's demonstration (*Dialogues Concerning Two New Sciences*, trans. by Crew and Salvio, pp. 20 sqq.) that the conception of a line as composed of an extended infinity of points requires also the existence of an infinity of gaps in the line.

For the Greeks the existence of an incommensurable length was demonstrated by its geometrical construction.

Eudoxus' solution[1] of the problem of the existence of incommensurable stretches consisted of postulating first the so-called Archimedean axiom: If L_1, L_2 are any two stretches then L can be added to itself, say n times, until $n \times L_1$ is greater than L_2. The effect of this axiom is to eliminate the possibility of the existence either of infinitely small or of infinitely large stretches. And in the next place the ratios of four stretches taken in pairs $L_1 : L_2$, $L'_1 : L'_2$ are equal if, for all integers m, n,

$$nL_1 > mL_2 \text{ implies } nL'_1 > mL'_2$$
$$nL_2 = mL_2 \text{ implies } nL'_1 = mL'_2$$
$$nL_1 < mL_2 \text{ implies } nL'_1 < mL'_2$$

In this manner the length of an incommensurable stretch, l say, is determined by a division of all the rational numbers m/n into three classes, viz. those which are less than l, those equal to l, those greater than l; and the second class contains either no, or exactly one, member.

Dedekind's Definition of Real Number

Dedekind's analysis of continuity is a natural extension of the mathematical method due to Eudoxus. The Dedekind definition of real number and a few of its consequences are discussed.

If Eudoxus' definition of the ratio of incommensurable lengths is used in conjunction with the criterion, already mentioned, of the existence of stretches, viz. that they can be constructed by the use of ruler and compass alone (for these were the only geometrical instruments used in Greek geometry) it is found that all those numbers 'exist' which can be obtained by applying to the rational numbers, any finite

[1] This account is based on Weyl (op. cit., p. 31).

number of times and in any order, the operations of taking the square root and the ordinary arithmetical operations of addition, subtraction, division, and multiplication; but numbers whose expression involves roots of higher orders cannot be obtained in this way. These so-called 'irrational' numbers however can often be given interpretations which make their existence intuitively plausible even at the stage at which the Eudoxian definition is regarded as satisfactory. Thus, for example, the cube root of a can be interpreted as the length of side of a cube of volume a. If the number a is a perfect cube, i.e. can be expressed as the cube of some rational number, then there is no difficulty. If this is not the case, e.g. if a is any prime number, any material cube of volume a could have a side whose length could not be expressed by the Eudoxian definition. Nevertheless, it is easily proved that any number of material cubes can be obtained both greater and less than a cube, whose sides have rational lengths and whose volumes differ as little as required from the volume of a. Thus, either a physical cube can be obtained whose volume is a, and this will mean that the Eudoxian definition is inadequate, or no physical cube can exist whose volume is a. The second of these alternatives is highly repugnant because it appears to involve gaps in the series of *rational* numbers which may be used to denote volumes, and arbitrary exclusion of rational volumes seems no better than arbitrary exclusion of constructible rational lengths. This type of argument applies of course to transcendental numbers as well as to irrationals of order greater than two. With the progress of mathematics the criterion of constructibility on a Euclidean plane begins to appear purely arbitrary; π soon comes to be as 'real' as $\sqrt{2}$.

It is at this stage that Dedekind produces his abstract definition of pure number. Suppose the rational numbers are divided *in any way* into two classes, L, R, say, such that

(i) all the members of L are less than all the numbers of R;

(ii) L contains at least one rational number;

(iii) all the rational numbers belong either to L or to R.

Any such division is called a *section* of the rational numbers. Then there are two different kinds of sections possible:—

(*a*) Either one number in L is greater than all the other numbers in L *or* one number in R is less than all the other numbers in R (*or* here excludes *and*).

(*b*) Or no number in L is greater than all the other numbers in L and no number in R is less than all the other numbers in R. In case (*b*) it is easy to show that the difference between two rational numbers, one chosen from L, the other from R can be made less than any number, however small, given in advance. L and R converge together but, as distinct from case (*a*) no *rational* number separates them. If (*b*) is the case, a 'real' number is said to be defined by the section, and is conceived of as a number, on the same level as the rationals which compose L and R, and filling the gap between them.[1]

The quotation from Dedekind already given on p. 87 leaves little doubt as to the status of his definition. It is essentially the definition of Eudoxus generalized to the extent that the criterion of constructibility has been dispensed with (for it does not matter how L and R are constructed provided they possess the three properties detailed above); but the existence of the real numbers is still based in some vague sense on geometrical intuition.

It is necessary to interpolate at this point a short account

[1] The principle was actually given the following geometrical form when first enumerated by Dedekind: "If all the points of a line are separated into two classes such that every point of the first class is to the left of every point of the second class, there exists one and only one point which produces this division of all the points into two classes and divides the line into two parts in this way."—(*Stetigkeit u. irrationale Zahlen*, p. 11, translated.) In the text the more arithmetical form which is now usual has been chosen but the two statements are essentially equivalent.

of the mathematical consequences of the Dedekind definition of real numbers :—

(1) The real numbers as defined by Dedekind leave no gaps in the field of geometric intuition. This can be expressed more exactly in two ways which are equivalent : if the Dedekind definition is applied to the *real* numbers, dividing them into classes L and R with the properties stated, no further numbers are obtained.[1] The alternative phrasing is as follows : no system of objects, S say, can be found obeying the axioms which the real numbers obey and containing all the real numbers as a subclass.[2]

(2) Any particular real number must be defined by actually stating the method or law for dividing the rational numbers into L and R and in all such cases it is true in general that the properties of the real number in question can be expressed in a rather more complicated fashion as properties of the rational numbers which are used in the definition of the specific real number. This is *not* the case however in certain very general theorems concerning the properties of functions (i.e. certain infinite collections of real numbers) ; these are the crucial cases where the contradictions inseparably connected with the extended infinite reappear. The case of the so-called 'theorem of the upper bound' is discussed below.

It is interesting to see that Dedekind himself appears to have been well aware of the provisional nature of his definition. He says : " Die Annahme dieser Eigenschaft [i.e. continuity as defined by the existence of real numbers] der Linie ist *nichts als ein Axiom* durch welches wir erst der Linie ihre Stetigkeit zuerkennen, durch welches wir die Stetigkeit in

[1] This statement is, of course, based on the ordinary realistic view of real numbers. It would not be true in the intuitionist mathematics because it would have no meaning to talk of two classes L, R of real numbers in this way.

[2] This is the formalist enunciation of the property and is used by Hibbert as a *definition* of continuity, i.e. by postulating the 'Vollstandigkeit' of his system of geometrical axioms he ensures Dedekind continuity : cf. his *Grundlagen der Geometrie*.

die Linie hineindenken. Hat überhaupt der Raum eine reale Existenz, so braucht er doch nicht notwendig stetig zu sein; unzählige seiner Eigenschaften würden dieselben bleiben, wenn er auch unstetig wäre. Und wüssten wir gewiss, dass der Raum unstetig wäre, so könnte uns doch wieder nichts hindern, falls es uns beliebt, ihn durch Ausfüllung seiner Lücken in Gedanken zu einem stetigen zu machen; diese Ausfülling würde aber in einer Schöpfung von neuen Punct-Individuen bestehen und dem obigen Princip gemäss auszuführen sein" (*Stetigkeit u. irrationale Zahlen*, p. 11 ff.).[1]

The Logistico-Mathematical Paradoxes

The paradoxes are classified according as they can or cannot be accurately expressed in mathematical symbolism.

The paradoxes and contradictions now to be described fall naturally into two classes :—

(*a*) Those which are due to the vagueness of words.

(*b*) Those which can be expressed in exact mathematical symbolism.

Of these two, (*b*) is by far the more important, for those features of (*a*) which do not reduce to (*b*) belong to a discussion of the limitations of any language which has evolved historically as an instrument for practical communication, rather than to a discussion of the foundations of mathematics.

[1] "The assumption that a line has this property [continuity defined by the existence of real numbers] is no more than an axiom by which the continuity of the line is recognized, or by which the line is conceived, in our thinking, to possess continuity. If space has any real existence at all, it need not be continuous, for innumerable properties would remain the same if it were discontinuous. And even if we were certain that space was discontinuous nothing could prevent us, if we pleased, from making it continuous by conceiving its gaps filled; such a process would consist of creating new points and would have to proceed in accordance with the above principle [i.e. the definition of real number]."

It will be well to quote a few examples of contradictions both of type (*a*) and (*b*). The letter which follows the description of the contradiction refers to this classification.

(1) *Weyl's contradiction* [1] (*a*).—"Some adjectives have meanings which are predicates of the adjective word itself; thus the word 'short' is short, but the word 'long' is not long. Let us call adjectives whose meanings are predicates of them, like 'short', autological; others heterological. Now is 'heterological' heterological? If it is, its meaning is not a predicate of it; that is, it is not heterological. But if it is not heterological, its meaning is a predicate of it, and therefore it is heterological. So we have a complete contradiction" (Ramsay, *Foundations of Mathematics*, p. 27).

(2) *The least integer not named in this book* [2] (*a*).—Some, but not all integers occur in this book, either as the corresponding cipher (the numbers at the head of each page for instance) or as an integer which satisfies a description. Only a finite

[1] *Vide Das Kontinum*, p. 2: "Ein Eigenschaftswort heisse autologisch wenn dieses Wort selber die Eigenschaft besitzt, die seine Bedeutung ausmacht; falls es sie nicht besitzt, heterologisch. Das Wort 'kurz' z.B. ist selber kurz (ein nur aus 4 Buchstaben bestehendes Wort wird man in der deutschen Sprache ohne Frage als ein kurzes zu bezeichnen haben) daher autologisch; das Wort 'lang' hingegen ist selber nicht lang, daher heterologisch. Wie steht es nun mit den Wort 'heterologisch'? Ist is autologisch, so hat es die Eigenschaft, die es aussagt, ist also heterologisch; ist es hingegen heterologisch, so hat es diese Eigenschaft nicht, ist also autologisch." Weyl's own solution is that the question whether the word 'heterologisch' is itself hetero- or autological cannot be given any sense.

[2] A refinement of a paradox given by Russell, cf. *Principia Mathematica*, p. 61, subheading (5). The paradox as given there is too vaguely stated to carry much conviction. The invention of contradictions is one of the lighter sides of mathematical logic. A good example is that of the barber in a village where all and only the men who do not shave themselves are shaved by the barber. If the barber does not shave himself, he is one of the men who are non-shavers and is therefore shaved by the barber, i.e. by himself. If, on the other hand, the barber does shave himself, he is one of the men who shave themselves, hence he is not shaved by the barber, i.e. he does not shave himself. Symbolically, the definition of the collection of men can be written $\hat{x}\{xSx \equiv \sim (bSx)\}$ ($S =$ shaves, $b =$ the barber), and the fallacy arises from the substitution of b for x in the defining equivalence. This illustrates the very important point that the mere formation of a definition of a class does not guarantee the existence (freedom from contradiction) of the class.

number of integers can occur in the book in this way for the number of words in the book is limited. Consider now *the least integer which does not occur in this book.* This phrase defines just one integer, hence by definition that integer occurs in this book, which is a contradiction.

(3) *The class of all classes which are not members of themselves* (*b*).—"Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, 'x is a w' is equivalent to 'x is not an x'. Hence, giving to x the value w, 'w is a w' is equivalent to 'w is not a w'." (*Principia Mathematica*, p. 60). Since classes are incomplete symbols, this contradiction can be translated in terms of functions and in this form the cause of the contradiction becomes very apparent. Let W be a function of functions X such that

$$W(X) = \sim X(X)$$

Substituting W, which is a function, for X in this equation we obtain $W(W) = \sim W(W)$, i.e. we cannot have $W(W)$. On the other hand if $\sim W(W)$, the same equation gives $W(W)$ hence in either case a contradiction.

(4) *Burali-Forti's contradiction* [1] (*b*).—(This paradox is inserted here on account of its mathematical importance and requires some knowledge of the mathematical theory of ordinal numbers.)

The following three theorems can be proved in the classical theory of ordinal numbers developed by Cantor.[2]

(i) Every well ordered series has an ordinal number.

(ii) The series of ordinals up to and including a given ordinal number, say O_1, has an ordinal number $O_1 + 1$.

(iii) The series of all ordinal numbers is well-ordered and hence, by (ii), has an ordinal number, Ω say.

[1] "Una questione sui numeri transfinite," *Rendiconti del circolo matematico di Palermo*, vol. xi (1897). See also p. 208 below.

[2] Cf., for instance, J. E. Littlewood, *Elements of Theory of Real Functions*, chapter 2.

But from (ii) the series of all ordinals including Ω has ordinal number $\Omega + 1$ which is greater than Ω. Hence Ω cannot be the ordinal number of all ordinal numbers.

Solution of the Paradoxes

They are produced by lack of indication of the field of variation of a variable in its determinative usage. Special technical devices (theory of types) are needed to ensure the appropriate restriction of fields of variation.

On examining the contradictions of which the above will serve as examples it will be seen that those of type (*a*) have a certain circularity in common with those of type (*b*), differing only in the relative inexactitude of notions like *description*, *adjective*, *occurring*, etc. This inexactitude consists of (*a*) type token ambiguity and (*b*) vagueness, which may be defined with reference to situations where it is impossible to decide whether the term in question applies or not.[1] When this inexactitude is eliminated by the use of more precise symbols, which may then be written in the form of the propositional calculus for convenience, it will be necessary to give rules stating which kind of symbol can stand as argument to a given functional symbol. For it has been seen that in our conception of the nature of the propositional function the argument forms part of the function-symbol; and unless the field of variation of the variable has been or is capable of precise definition, the meaning of the propositional function will be indeterminate. It has been already explained that a variable may be correctly used (determinative usage) to obtain a more precise description of the field of variation when that field is initially unknown.

The essential feature of the fallacies committed in the four contradictions given above is therefore as follows: the

[1] *Red*, for instance, is a vague concept because colours may be presented for which it is impossible to say whether they are red or not, i.e. for which the question "Is this colour red?" begins to lose meaning. cf. the author's "Vagueness," *Philosophy of Science*, iv, 427-455.

field of variation of the variable involved is taken to be *all* the members of a certain class which proves to be wider than the variable in its determinative usage permits. The example of Weyl's contradiction will illustrate this: if heterological is heterological this means, by definition, that its meaning is not a predicate of the word itself, i.e. its meaning is defined in terms of its meaning and we are no nearer understanding what this meaning is. Thus the word *heterological* cannot be part of the field of variation of the argument of the function *heterological*. The fallacy consists of assuming that the field includes *all words*.

Russell's solution of this difficulty is to adopt the principle that no function can be a value of its own argument. This restricts the field of variation of every variable and eliminates the contradictions. The effect of the principle is to segregate functions into distinct types or levels. No function can take a function of the same or higher type as level. That is the first part of the *Theory of Types*.

Note on Types and Orders

The method adopted in *Principia Mathematica* for restricting fields of variation involves a hierarchy of types and a subsidiary hierarchy of orders. The latter is untenable and must be abandoned: two methods for dispensing with it are noted.

The segregation of propositional functions into various levels or types according as they are functions of individuals or functions of functions of individuals, etc., is a classification too crude for the *Principia* reduction of mathematics to logic, and it can be shown that the principles on which that classification was based demand a further sub-classification of functions of the same type or else a radical alteration in the whole notion of propositional functions. For the paradoxes which the first part of the theory of types was designed to remove are all ultimately based on the employment of variables with indefinite fields of variation, i.e. fields so defined that crucial cases arise in which it is impossible to say whether a given individual belongs to the field without *previously* knowing all the members of that field. That indefiniteness of this kind is a consequence of permitting circular definitions of a propositional function in terms of itself is obvious; but segregation into types, while obviating this specific possibility of circularity, permits others of the same kind and with consequences as disastrous. For expressions such as $(\phi) f(\phi, x)$, where ϕ is a variable propositional function of type one and f some constant propositional function of type two, define a function of individuals (therefore of type one), by means of a totality of functions of an individual. Hence the field of variation of a variable propositional function of type one is viciously indeterminate and paradoxes will recur unless functions of the same type are subdivided into further levels, which may be termed *orders* to distinguish them from types.

Two attempts have been made to define orders in the first and second editions respectively of the *Principia Mathematica*. They must be very briefly sketched at this point. In the first attempt *individuals* are defined as things which are neither functions nor propositions, and *matrices* as values of functions in which all the variables are real—e.g. $\phi(x, y)$ or $f(x, y, z)$, where x, y, z are variable individuals, are *first order* matrices. Functions obtained by quantifying first order matrices, i.e. by converting some of the variables in such matrices into apparent variables, are termed first order functions (of individuals); *second order* matrices are matrices some of whose variables are first order functions, *second order* functions are derived from them by quantification, and so on, nth order functions being defined by (mathematical) induction.

Objections to this subsidiary hierarchy can be raised on the score of the immense complexity thereby introduced and to the extreme difficulty of discovering the order of specific propositional functions. The latter of these is the crucial criticism arising from the fact that the principles implicit in the notation of propositional functions according to which functions of continually increasing complexity may be constructed (e.g. quantification of variables, replacement of variables by constants, identification of various variables, etc.) do not coincide with the simple principle for constructing series of matrices (and hence functions) of increasing orders.

An attempt is made in the second edition to remedy these defects by systematizing the principles for constructing propositional functions; the form of propositional functions of lowest type is specified more exactly, operations which may be applied to them in order to produce functions of greater complexity are restricted, and the definition of matrices is modified. Fundamentally, however, these alterations do not save the second part of the theory of types from the criticisms made above, and most writers have rejected it while preserving the distinction of types.

It was implied above that an alternative to the introduction of orders would be alteration in the conception of propositional function; and there are two entirely different methods which have been pursued with this purpose. If a propositional function is regarded extensionally as a collection of the arguments (or ordered pairs of arguments in the case of a two-termed function) which satisfy it, distinctions of order between formally equivalent functions appear as differences merely of expression and not of meaning or reference of the two functions; such a view if pursued consistently appears to involve the identification of formally equivalent functions and reasons have been given above for rejecting it. The best exposition of this type of solution may be found in Ramsey's *Foundations of Mathematics*; Carnap's thesis of extensionality is based on similar considerations and has the same consequences.

The remaining of the two methods mentioned is to adopt the intensional definition of propositional function given on p. 57 above, which automatically ensures the absence from circularity required to avoid all paradoxes which segregation into types eliminates. For, as previously stated, the essence of the paradoxes lies in the indeterminacy of the notions used and especially in the notion of propositional function; precision in the definition of the latter will eliminate the paradoxes. The theory of orders in its second form appears to be an essay towards such precision and Chwistek's *Theory of Constructive Types* (see p. 135) is a more elaborate attempt of the same kind.

It will therefore be assumed in the sections which follow that, while distinctions of level of *some* kinds are necessary and the hierarchy of types is both valid and useful, the hierarchy of orders is not to be adopted in the form chosen in *Principia Mathematica*. The one term *type* will be here used to denote the kind of distinctions of level which may ultimately be necessary; and no attempt will be made to distinguish between orders and types.

Before proceeding to a detailed examination and criticism of the theory of types we must attempt to show the connection between the so-called extended infinite and the contradictions already mentioned.

Connection Between the 'Extended Infinite' and the Paradoxes

The use of the 'extended' infinite is equivalent to a confusion of types. Difficulties inherent in an extended infinity or geometrical continuum are therefore reproduced in the theory of propositional functions and their correction tends to destroy the possibility of adequately symbolizing, by the propositional calculus, of the mathematical theory of functions.

First, the notion of the 'extended infinite' must be made a little more precise. It has already been shown (p. 55), as is indeed obvious, that the list of pairs of numbers consisting on the one hand of the field of variation of the argument and on the other hand of the corresponding values of the function can be completely enumerated only if the specific field of variation contains only a finite number of members. And, in general, any specific infinite 'list' of numbers can be given only by supplying a law which will successively produce each member. A vague description of what occurs when the infinite is regarded as 'actual' is that infinite lists or collections are considered to be of precisely the same nature as finite ones; and the impossibility of enumerating such infinite lists is regarded as in some way psychological, a feature of man's limitations in the presence of reality. Belief in the reality of infinite collections shows itself in the writings of those who share it in two ways:—

(*a*) in certain metaphysical pseudo-propositions of the type "infinite collections are real". Such propositions prove to be incapable of either verification or disproof; and it appears to be impossible either to analyse or describe the concept of 'reality' involved in such statements.

(*b*) in the manner in which the corresponding symbols are used in non-metaphysical, i.e. scientific or mathematical propositions, capable of disproof or verification either by the methods of experimental science or of mathematics. In accordance with what has been said above as to the definition of an infinite list requiring a *law* concerning the manner in which members are obtained, it always happens that *symbols defining infinite classes are of a higher type than those defining finite classes.* This is true whether the symbols of the propositional calculus and Russell's definition of the type of a symbol are used, or any other alternative array of symbols and symbolic conventions; for the necessity for dividing symbols into types is based on the need (1) for specifying exactly the field of variation of variable symbols, and (2) for defining new symbols in terms of old, and therefore recurs, either explicitly or by implication, in all systems.

Hence the effect under head (*b*) of a belief in the actual infinite is that, since infinite lists or collections are considered to be on the same level as finite ones, the corresponding symbols are treated as if of the same type.

We are not concerned here with that part of the use of and belief in the actual infinite which fall under head (*a*) above, for these elements are in their nature unsuited for discussion. So, restricting attention to (*b*), the above discussion of the use of the 'actual infinite' can be concluded by saying that its effect is essentially that *symbols of different types are treated as having the same type.*

If this is a correct account, it is to be expected that (1) the modern theory of functions will actually contain confusions of type of exactly the kind which occur in the paradoxes given here and (2) that Russell's theory of types, by removing this confusion and compelling the distinction between types to be rigidly observed, will accomplish too much and will destroy the validity of some theorems in the theory of functions which have been accepted by mathematicians. The dilemma is indeed a more formidable one than this formulation suggests, for the identification of different types happens continually in mathematics whenever formal analogies between symbols of different lines of complexity are discovered; Russell's theory of types is particularly stringent and makes the formulation of all such cases a matter of great difficulty.

It is quite easy to show that confusions of type are common in the theory of functions. Dedekind's definition of the real number (p. 94) at once produces examples.[1]

[1] I follow here the argument of Weyl, *Das Kontinuum*, pp. 19 ff.

Confusion of Types in the Theorem of the Upper Bound

A specific and important example of confusion of types in pure mathematics is here demonstrated.

In order to simplify the account somewhat we may assume that we are dealing with variables x, y, z, and undetermined constants a, b, c, which denote rational numbers, and in addition certain propositional functions, $F(x)$, $G(x)$, etc., in which the rational numbers occur as real variables; the fact that x, y, z, etc., are themselves functions or incomplete symbols of a high type in the exposition of *Principia Mathematica* will not affect the discussion.

It has been seen that a real number is defined by dividing the rationals by a 'Dedekind section' i.e., in the present symbolism, any specific real number must always be defined by some propositional function, $F(x)$ say, such that F has the properties expressed by headings (i), (ii), (iii) on p. 95. For example $x^2 \leqslant 2$ will be such a function defining the real number $\sqrt{2}$. For we have only to put in L (p. 94) the rational numbers which make the statement $x^2 \leqslant 2$ true and in R the numbers which make $x^2 \leqslant 2$ false.

Then (i) every L is less than every R, for if $a^2 \leqslant 2$ and $2 < b^2$, a^2 must be less than b^2 and hence $a < b$;

(ii) contains at least one rational, viz. $1/2$;

(iii) all the rational numbers obviously belong either to L or to R, since for each x, $x^2 \leqslant 2$ is either true or false; if the first, x belongs to L, if the second to R. Let us call the three conditions (i), (ii), (iii), taken together, C for convenience.

By this method of definition, there will be a (many-one) correlation between those functions $F(x)$ which satisfy C and the real numbers. Now it has been seen that, unless some restriction is placed upon the method of formation of the functions $F(x)$, contradictions will occur, i.e. functions can be produced which satisfy C but whose employment is self-contradictory.[1] Let the functions $F(x)$ which are used to define the real numbers be termed functions of type one. Some 'real numbers' are defined in terms of real numbers already defined; an important case is that of the so-called upper bound. Given any collection of real numbers, i.e. the real numbers defined by all those functions of type one which satisfy C, and in addition some specific D which defines the collection, the upper bound U is defined as the real number produced by a Dedekind section (L_1, R_1) such that any rational number x belongs to L_1 if and only if it belongs to the L class of one of the $F(x)$ which satisfies D.[2] It is easy to show that such a bound always exists if all the real numbers are less than some given number.

The number U however has been defined by a propositional function *into which functions F enter as variable*, i.e. U is defined by a function of type two. In mathematical text-books, however, U is treated as of exactly the same kind as real numbers such as $\sqrt{2}$ defined directly in terms of the rationals; this is an identification of types of exactly the kind that Russell's theory of types prevents.

It should however be noticed that this result does not prove that the mathematicians procedure is *incorrect* (as Weyl appears to suggest by the use of 'vicious circle' to describe the situation) for all that has been shown is (*a*) the use of symbols requires *some* conventions as to type and (*b*) if Russell's theory of types is correct the mathematician's construction of the upper bound is definitely incorrect. On the other hand (*c*), when specific real numbers are used, it

[1] e.g. let $F(x) . = . x$ is less than the least integer not named in this book (p. 98). $F(x)$ satisfies C but is self-contradictory.

[2] In the language of *Principia Mathematica* this can be simply expressed by saying that L_1 is the sum class of all the classes L corresponding to the real numbers of the particular collection considered.

is nearly always possible to give equivalent functions of type one to define any further real numbers (upper bounds, limits, etc.) which may be required—so that only the most generalized theorems of the theory of real functions fall under this criticism: and (*d*) circularity need not be 'vicious'—it might be possible to invent consistent conventions of type which permitted circularity of certain specified kinds and in particular such as to permit the construction of the upper bound of any collection of real numbers.

The Axiom of Reducibility and the Logistic Definition of Real Number

Dedekind's definition of real number is based on vague geometrical intuitions and is therefore unsatisfactory; but the *Principia* substitute needs axioms of infinity and reducibility whose validity is doubtful.

The essential defects of Dedekind's definition of real number are (*a*) that the evidence for the correctness of the definition rests on geometrical intuition of the relations between ideal points and lines in a specific geometry whose selection for this purpose may be attributed either to historical reasons or to an (alleged) necessary connection between Euclidean geometry and the relations of apprehended sense-data. Since Euclidean geometry is one only of many that can now be constructed the evidence for the existence of real numbers requires to be of a nature at once more general and more reliable.

(*b*) The notion of the 'existence of a real number' is vague and requires further analysis. For example, it becomes a matter of critical importance to determine whether, and in what circumstances, the existence of a real number implies more than the presence of some method for calculating its value to any required degree of approximation.

The truth of (*a*) clearly implies the truth of criticism (*b*); those who would defend Dedekind's definition—a group which no doubt includes a great many expert pure mathematicians—would need to reject criticism (*a*).

Thus it might be argued that the 'existence of a real number' is not synonymous with the 'knowledge of a necessary and sufficient criterion' or even with the *existence* of a necessary and sufficient criterion for the existence of a real number. Hence, although the Dedekind definition does in fact supply a necessary and sufficient criterion for the existence of *some* real numbers (namely, the possibility of separating the rationals into the classes L and R described above), it would be urged that real numbers may exist which are incapable of definition.

There are at least two views that might be held as to what is meant by the existence of a real number :—

(1) (Realist argument). It might be urged that the Dedekind criterion for the existence of a real number cannot be self-contradictory; that the presence of a sufficient and/or necessary criterion for the existence of a real number is not the same as the existence of a real number; and that the existence of all the real numbers used in mathematical analysis is guaranteed by evidence based on geometrical intuition. This argument is based on the supposed identity of the symbol *exists* in two such sentences as $\sqrt{2}$ *exists* and *The King of England exists*, i.e. on a confusion between two senses of existence, the first that in which a number can be said to exist and the second that in which a person can be said to exist. This view is therefore a mistaken one.

(2) (Neo-Machian standpoint). Existence of a real number is synonymous with the presence of a necessary and sufficient criterion. This view also requires a revision of the Dedekind definition in order to avoid contradictions.

Thus Dedekind's definition cannot be regarded as a final solution. This fact was clearly recognized among others by Russell whose emendation consisted essentially in defining the class L itself (cf. p. 94) as the real number, i.e. the real number was defined as the class of all the rational numbers less than it.[1]

This definition apparently eliminates the difficulty about the meaning of the 'existence' of a real number for in Russell's own language it 'constructs' the real number instead of 'postulating' it. The difficulties associated with existence reappear however almost as obstinately in the logistic scheme in the following forms:—

(1) the axiom of reducibility.

(2) the axiom of infinity.

The Axiom of Reducibility[2]

In accordance with the logistic definition of real number just described a real number is a class of rational numbers and an upper bound (p. 108) is a class of real numbers; hence the upper bound of a set of real numbers is a real number of higher type. Thus there must be infinitely many different types of real numbers.

The solution of this dilemma in *Principia Mathematica* is to postulate that each propositional function of any type whatsoever has some propositional function of type one formally equivalent to it. This effectually destroys the segregation of types without reproducing the contradictions as might at first be supposed. For the contradictions depend on the *meaning* of the propositional functions in question, whereas for mathematics only the truth values of propositions matter so that any propositional function can be replaced by any formally equivalent propositional function.

[1] *Vide* Russell, *Intro. to Math. Phil.*, ch. vii, for a detailed definition.
[2] *Vide Principia Mathematica*, vol. i, pp. 55–60, pp. 160–7, and *12.

This principle of the existence of propositional functions of lowest type formally equivalent to any propositional function was termed an axiom by Russell presumably because there appeared to be no method of proving it; but the difficulty facing all subsequent commentators has been not so much to decide whether it is true as to understand what is meant by asserting the existence of propositional functions.

The Axiom of Infinity [1]

Russell's definition of integral number is based on the existence of a sufficient number of propositional functions with certain properties; to put the matter very crudely, if there are only a finite number of propositional functions, only a finite number of integers will exist. Thus an 'axiom of infinity' is required, postulating the existence of infinitely many propositional functions.[2] Here again the stubborn difficulty is to understand what can be meant by the *existence* of the required propositional functions.

Arguments for the Axiom of Reducibility

Russell's arguments in favour are criticized but conventions are suggested which may remove some of the objections to the axioms of reducibility and infinity.

Russell's arguments in favour of the axiom of reducibility in the first edition of *Principia Mathematica* (vol. i, pp. 55–60)

[1] *Principia Mathematica*, *125 and vol. ii, p. 183.

[2] It is by no means obvious that the *Principia Mathematica* statement of the axiom occurring at an advanced stage in the architectonic superimposition of definitions does in fact reduce to an axiom of the nature stated in the text above; but the detailed analysis required to demonstrate this would be out of place here. It may, however, be noticed that the *P.M.* form of the axiom could be simplified and that the existence of *one* relation conforming to certain specified conditions would probably be sufficient. The relation in question must be a one-one relation whose converse domain is strictly contained in its domain, for a class which can be put into a one-one correlation with a subclass of its members must have an infinity of members.

amount to the contention that "the reason for accepting any axiom, as for accepting any other proposition, is always largely inductive, namely that many other propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it". The first part of the quotation appears to be of doubtful validity since in the logic of *Principia Mathematica* a false proposition implies every proposition, so that all the 'indubitable' propositions could very well be deduced from the axiom if it happened to be false; and the remainder of the quotation tries to justify expediency by an appeal to the truth of unverified hypotheses.

The fundamental reasons for introducing the axiom are clearly indicated in another statement of Russell's: "The axiom of reducibility is introduced in order to legitimize a great mass of reasoning in which, prima facie, we are concerned with such notions as 'all properties of *a*' or 'all *a*-functions' and in which, nevertheless, it seems scarcely possible to suspect any substantial error" (ibid., p. 56). Which means that the axiom is introduced in order to be able to make precisely those general statements involving the term *all* which the theory of types forbids.

To state the axiom as in *Principia Mathematica* in the form that every function is equivalent, for all values of its argument, to some function of the lowest type is misleading, and has led to some unjustified criticism of the theory of types. For the enunciation of the axiom itself appears to sin against the theory of types by mentioning 'all propositional functions' and thus invites adverse criticism which seeks to establish self-contradiction in the notion of a hierarchy of types. This misunderstanding is produced by the insufficient discrimination made in *Principia Mathematica* between those theorems which are part of the deductive system and those which are directions for and restrictions on the use of symbols in the deductions. The theory of types belongs to the latter class and consists essentially of (insufficient) directions for the unambiguous use of propositional function-symbols; it is a device analogous to the creation in mathematics of new symbols by the attaching of indices to a stock of symbols insufficient by themselves to represent adequately and unambiguously the relations of the field investigated. Such supplementation is necessary only when the conventions of significance, i.e. the rules according to which the symbols may be combined, are insufficient to ensure unambiguity and consistency. Of such conventions, some are necessarily determined by the choice of symbols (visible or tangible) to be used, others are implicit in the silent agreement of those who employ the system of symbols; while the remainder need to be explicitly stated. No difference of principle can, however, be found in the last group and, in theory at least, the need for stating such conventions can always be avoided by using new symbols of a higher degree of multiplicity.

Thus, for example, the theory of types could be entirely eliminated from the logistic system by using instead of marks wooden rings to represent propositional functions. The requisite conventions of significance might be of the following nature: Functions of the same type have equal radii and the argument to any function is a ring of smaller radius fitting tightly into it. The theory of types would then be *shown* by the fact that no ring could fit into another of equal or smaller radius.

On applying similar considerations to the axiom of infinity it appears that the simplest interpretation of the latter is to regard it as a rule for constructing as many new symbols, as required. Thus in the system of ring symbols described the axiom of infinity would be replaced by an understood convention for constructing an unending series of symbols, e.g. by allowing rings of a certain length of radius to have thickness, one, two, three, . . . units.[1] In order to be valid in the logistic scheme the axiom of reducibility should analogously be capable of reduction to a statement about the manner in which symbols could be constructed in that system.

Axiom of Reducibility Equivalent to the Assertion of the Existence of *c* Propositional Functions

A mathematical interlude in parenthesis to the main argument.

It has thus been seen to be possible to eliminate both a theory of types and an axiom of infinity from the logistic scheme; the axiom of reducibility however presents more formidable difficulties into whose analysis we must now enter.

(1) It is not possible to transform the axiom of reducibility by choosing an appropriate system of symbols of the requisite multiplicity. For, just as the function of the axiom of infinity is to furnish an enumerable or *countable* infinite and thus to ensure the existence of the natural numbers, so part of the function of the axiom of reducibility is to ensure a supply of propositional functions of cardinal number c, i.e. capable of being put into one-one correlation with the parts of a continuum.

Thus it can be shown that if the deduction of *Principia Mathematica* ensures the existence of all the real numbers, it must postulate the existence of a set of predicative functions, (i.e. of type one), no two having the same extension and sufficiently many to be put into one-one correspondence with the points of the continuum; or, more concisely, there must be c predicative functions with different extensions.

[1] This interpretation of the axiom would in turn be open to objections, and is not intended as a final analysis of the nature of infinity.

For a real number is a class of rationals; a rational is a class of couples of cardinal numbers; and a cardinal number is a class of classes whose members can be put into one-one correspondence. Hence a real number is a class of type four.[1] Since all the real numbers are different by hypothesis, there must be at least *c* propositional functions of order four, no two of which have the same extension. But by the axiom of reducibility each of these functions has a predicative function which is equivalent to it. Thus there are *c* predicative functions with different extensions.

Having thus shown that the statement of the axiom of reducibility taken in conjunction with the other axioms alone implies the existence of *c* predicative propositional functions it is now easy to see conversely that, without the axiom of reducibility, it would be impossible either (*a*) to obtain the *c* predicative functions by construction or (*b*) by appeal to empirical fact.

For (*a*) no constant propositional functions are used in the *Principia Mathematica* definitions, all the integral numbers and then by successive stages the rational and real numbers being defined with the help only of the propositional function *identity* which holds between x and y when they satisfy the same predicative functions. All other functions must be constructed from the primary one of identity by the use of a finite number of logical operations (quantification, negation, etc.). Thus even with the axiom of infinity all that can be obtained is an enumerable infinity of enumerable infinities, that is, an enumerable infinity,[2] and (*b*) no empirical evidence can be given of the existence of infinitely many different *constant* propositional functions.

[1] Actually, however, owing to the use of the axiom of reducibility and to refinements in the definition, real number as defined in *Principia Mathematica* (*310.01) is of higher order. This, however, does not affect the argument.

[2] A well known result in the theory of cardinals. This leads to the pretty paradox that even *with* the axiom of reducibility there will always be propositional functions which cannot be constructed.

Yet, the whole of the correctness of the *Principia Mathematica* construction depends precisely upon the unverifiable *existence* of these *c* predicative functions. Crudely stated: if there are fewer than *c* predicative propositional functions some of the numbers considered distinct by mathematicians will really be identical!

Other Criticisms of the Axiom of Reducibility

The attempts hitherto made to prove the axiom of reducibility a contingent proposition are fallacious.

Criticism of the axiom has usually been devoted to its lack of evidence, and a certain amount of work has been done to investigate whether it is a contingent proposition.

Ramsey's [1] attempted proof that the axiom is contingent, and Waismann's [2] elaboration of that proof are both fallacious. The method used by them consists in making certain assumptions (*a*) concerning the number of individuals in the universe, (*b*) concerning the number of predicative propositional functions, and (*c*) the number of predicative propositional functions which are satisfied by each individual. If, in such a universe, a non-predicative propositional function can be constructed and shown to be equivalent to no predicative propositional function, the axiom of reducibility would be false in that domain. If this could be proved it is held that the axiom of reducibility would be an empirical proposition. The mistake made in the proofs referred to above consisted in neglecting to observe the *necessary* conditions which predicative propositional functions must obey, e.g. if f is a predicative propositional function so is $\sim f$; if f and g are so is $h(x) = f(x) . g(x)$ *Df*. Thus statements (*a*), (*b*), (*c*) above must conform to these conditions.

[1] F. P. Ramsey, *Foundations of Mathematics*, p. 57.
[2] F. Waismann, "Die Natur des Reduzibilitätsaxiom," *Monatshefte für Mathematik und Physik*, vol. xxxv, 1928.

In accordance with our previous discussion, since the axiom postulates the existence of *c* predicative functions of various extensions and the number of constructive operations that the predicative propositional functions can undergo is finite, it would appear to be possible, in any case where there are only enumerably many predicative propositional functions, to construct a non-predicative function which has no equivalent predicative function. The discovery of the necessary and sufficient conditions for the axiom of reducibility to be true in a domain of one-valued predicative functions is however difficult and has so far not been accomplished.

Before concluding this part of the investigation by a summary of the criticism against *Principia Mathematica*, a few sections will be devoted to reporting on improvements of that work. The authors considered are (the late) F. P. Ramsey, H. Weyl, (the late) L. Chwistek, and L. Wittgenstein.

F. P. Ramsey

This, and the three sections which follow, are a report on various attempts to remedy the worst defects of *Principia Mathematica*. Ramsey's principal contribution was the effort to dispense with an axiom of reducibility by using functions defined in terms of truth values with a minimum of specific reference to symbols.

The feature of Ramsey's work which is most important for present purposes is his attempt to eliminate the need for an axiom of reducibility in the logistic structure. In *Principia Mathematica*, as previous discussion has indicated, there is to be found a prolonged compromise between an early realist attitude towards classes, and a later theory which regards them as incomplete symbols and reduces all statements concerning their existence to the assertion of specified sets of correspondences between constructed symbols. Either attitude, consistently elaborated, can culminate in a system containing no axiom of reducibility. The second, however, requires meticulously precise indication of the denotation of the term *symbol*. For a system of symbols has significance only in the process of being used by persons and its meaning is derived from the information which those who use it intend to express. All such words as *variable, symbol,* etc., involve mental dispositions or states in their definition and it is the ineradicable ambiguity and vagueness of the names for such states which ultimately necessitates a step by step constructive definition of terms like function whose reference to them is apparently most indirect. The latter procedure leads naturally to a theory such as that developed by Chwistek or Weyl. Ramsey, on the other hand, in his earlier work at least,[1] adopted a thoroughly realist attitude towards classes,

[1] Through the kindness of Mr. R. B. Braithwaite I have had access to some unpublished work by Ramsey, written shortly before his death, which indicates that he was developing a view of mathematics similar to that of Brouwer.

conceiving them to exist irrespective or independently of the possibility of their definition. His definition of *truth functions of propositional functions* can be most conveniently taken in stages. First, he says, a propositional function is a symbol or expression (*Foundations of Mathematics*, p. 8). An *atomic proposition* is one "which could not be analysed in terms of other propositions and could consist of names alone without logical constants" (loc. cit., p. 5), and an atomic fact is the fact which is expressed by such a proposition if the proposition is a true one. A truth function of propositions is, in accordance with our own definition (cf. p. 67), one whose truth or falsehood depends only on the truth or falsity of the propositions which are its arguments. But Ramsey asserts that a truth function may have an infinite number of arguments (loc. cit., p. 7). A function of individuals is *atomic* if all its values are propositional functions. And the definition of *truth functions of propositional functions* is "Suppose we have functions $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc., then by saying that a truth function $\psi(\hat{x}, \hat{y})$ is a certain truth function (e.g. the logical sum) of the functions $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc., and the propositions p, q, we mean that any value of $\psi(x, y)$, say $\psi(a, b)$, is that truth function of the corresponding values of $\phi_1(a, b)$, $\phi_2(a, b)$, etc., and the propositions p, q, etc." (p. 38). Finally "a predicative[1] function of individuals is one which is any truth function of arguments which, *whether finite or infinite in number*, are all either atomic functions of individuals or propositions" (p. 39).

The range of predicative functions thus defined is claimed to include all those occurring in *Principia Mathematica*; for the turning of real variables into apparent by the application of quantifiers is conceived of as either an infinite logical product or infinite logical sum of certain truth functions of

[1] This is, of course, not the sense in which *predicative function* is used in *Principia Mathematica* and in this book. Ramsey replaces that use of *predicative* by elementary.

propositional functions as above defined. Hence all the propositional functions of individuals, for example, belong to the same range, and the second part of the theory of types is unnecessary.

The following comments may be made on the above scheme: Ramsey says "by a propositional function of individuals we mean a symbol" (loc. cit., p. 35) and again "functions are symbols".[1] How then can a symbol have an infinite number of arguments? Yet Ramsey's notion of predicative function is useless unless an infinite number of arguments are allowed. His definition, as he himself says, is essentially dependent on the notion of a truth-function of an infinite number of arguments; "if there could only be a finite number of arguments our predicative functions would be simply the elementary[2] functions of *Principia*" (loc. cit., p. 39).

The only explanation possible is to regard the *fact* corresponding to any general or instantial proposition (i.e. a proposition containing a quantified apparent variable) as composed of an infinity of atomic facts and to regard these atomic facts as arguments to the general fact. Ramsey himself makes very clear that this is his position, thus all our criticisms of positions which accept the 'actual' infinite will apply with maximum effect to his exposition.

[1] The sense in which symbol is to be understood here is that in which 'symbol' is the determinable of which *word*, *phrase*, *sentence*, etc., are determinates.

[2] i.e. what are in this book called *predicative functions*, see footnote on previous page.

Note on the Thesis of Extensionality

A dogma of the logical positivists is examined.

It is instructive to compare Ramsey's theory of extensional function with the thesis of extensionality (Extensionalitäts-these) held by Carnap and others of the Austrian positivists.[1] It is asserted, on the basis of a distinction between the 'Sinn' and 'Bedeutung' of all symbols, that *all* functions of propositional functions are extensional, i.e. that any true statement involving a propositional function remains true when a formally equivalent propositional function is substituted. The terms employed in making the distinction referred to are ambiguous; sometimes the antithesis of the two aspects appears to correspond to that between 'connotation' and 'denotation', e.g. $4 + 3$ and $5 + 2$ are said to have different senses (Sinn) but the same meaning (Bedeutung). For our purpose however it is unnecessary to analyze the distinction in detail for, restricting our discussion to the Bedeutungen of propositional functions, we may adopt the methods of the logistic calculus, treat this phrase as an incomplete symbol, and discover its import by eliminating it from the contexts in which it occurs.

Cases which seem to disprove the thesis of extensionality are statements such as *x is a man has seven letters* which become false when the formally equivalent function *x is a featherless biped* is substituted for *x is a man*. The answer made is that the statement in question is not really about the propositional function *x is a man* but about the sign by which it is expressed, and the whole thesis hinges on the

[1] Cf. Carnap, *Der Logische Aufbau der Welt*, p. 62.

question as to when a statement is really 'about' a propositional function. The answer supplied by Carnap in *Der Logische Aufbau der Welt* can be disentangled as follows :—

(1) Only extensional statements are about propositional functions themselves.

(2) Extensional statements are about the 'Bedeutungen' of propositional functions.

(3) The Bedeutung of a propositional function is the class of formally equivalent functions.

Hence no true statement can be made 'about' any propositional function which is not equally true of every equivalent propositional function, i.e. there is no method of distinguishing formally equivalent propositional functions; formally equivalent propositional functions are identical.

The same result can be obtained otherwise; α is a propositional function of one variable means α has an argument place and if any x is substituted in that place a sentence is obtained. Now the Bedeutung of a sentence (in the indicative mood) is its truth value so that the identity of two propositional functions must reduce to the identity either of the Bedeutungen or else of the Sinne of the corresponding sentences; on the first alternative formally equivalent functions are identical and on the second the thesis of extensionality is incorrect.[1]

Thus Carnap's position reduces to the use of Ramsey's extensionally defined propositional functions and the same criticisms will apply to both.

[1] It may, however, be the case, even if the thesis of extensionality is incorrect, that mathematics treats only of extensional statements.

H. Weyl

In sharp contrast to Ramsey, Weyl attempted to systematize the principles by which symbols, especially propositional functions, are constructed. His method is summarized and its consequences noted.

Weyl is now one of the most famous supporters of the intuitionist philosophy of mathematics, but *Das Kontinuum*, one of his earlier works on the nature of mathematics, is a very ingenious attempt to construct with only a finite number of principles of construction a continuum of the real numbers required in mathematical analysis. His results can be adapted to form part of the logistic construction of mathematics, and are especially important for that purpose because no axiom of reducibility or equivalent axiom is used.

Weyl was concerned to remove the suspicion of vicious circularity attaching to the mathematical theory of functions and sought to attain this by using as propositional functions only such symbols as can be manufactured by a finite number of applications to symbols already defined of the principles described below. Since he restricted his attention to the definition of real numbers in terms of the natural numbers, he began his analysis at a stage where it is assumed that the following symbols have already been introduced either as undefined, or else defined by means of axioms (p. 47), or by definitions in use in terms of other symbols previously introduced as in the logistic scheme :

(i) The natural numbers, which are to function as 'individuals' in the present account.

(ii) A few constant propositional functions which variables from (i) can satisfy. These are certain mathematical relations such as *is greater than, is the product of*, etc., and *identity* (symbolized as a relation).

(iii) The logical operations of *conjunction, negation*, and *disjunction*, together with *quantification, the latter to apply only to variable natural numbers.*

(iv) The following two operations which derive propositional functions having a smaller number of arguments than the functions from which they are derived: (*a*) the identification of variables, as when the function of two arguments $R(x, y)$ becomes the function of one argument $R(x, x)$, (*b*) the substitution of constants for variables, as when the function of three arguments $S(x, y, z)$ becomes the function of two arguments $S(x, y, a)$. Only natural numbers may be substituted in (*b*).

(v) As in the logistic scheme, classes may be defined as incomplete symbols with the usual properties; to every function $F(x)$ there corresponds a class F; the expressions *a has the property F*, $F(a)$, or *a belongs to the class F* have the same meaning; two functions $F(x)$, $F'(x)$ define the same class when and only when every object which satisfies $F(x)$ also satisfies $F'(x)$ and vice versa. Similarly 'two dimensional sets', i.e. classes of ordered couples, are defined.

(vi) Functions can be formed with these new categories of objects (classes, etc.) as variables; hence every function must indicate the category to which its variable belongs.

So far what has been defined is a 'restricted' calculus of propositional functions. Next follow some special principles for constructing new symbols which are easiest to explain by exemplification.

(vii) $R(uv/xyz)$ is a propositional function with five arguments of the same category, where x, y, z are distinguished from the rest by being placed to the right of the stroke and are called free variables while u, v are called dependent variables. Then for any given values a, b, c, of x, y, z, there is a two-dimensional set consisting of the pairs of values of u and v which satisfy $R(uv/abc)$. This set, ϕ_{xyz} say, is a variable class depending on the choice of a, b, c and is introduced to take the place of the *mathematical function*.[1] For example, suppose $R(uv/x)$ is $u - v = x$, all the variables being positive integers; the principle allows the formation of a derived function, $A(x)$ say, which correlates to each positive integer x the class of pairs of positive integers whose difference is x.

(viii) A principle of extended substitution: variable classes, as defined in (vii) may be substituted for variables of appropriate categories to form new functions, e.g. from two propositional functions $R(uv/xyz)$, $S(xwU)$ where all the letters except R and S are variable and U is of the category *class of ordered couples*, we can successively form ϕ_{xyz} from R (as in (vii), and then $S(x, w, \phi_{x,y,z})$, a function of four arguments.

(ix) A principle of iteration; first in a narrow form and immediately extended; let $R(xx'/X)$ be a propositional function, where X is a variable class of couples of entities of same category as x and x'. Forming the class $\phi(x)$ as before, and using (viii), the function $R_2(xx'/X) = R(xx'/\phi(x))$ is obtained.

Similarly we can proceed to define

$$R_3(xx'/X) = R_2(xx'/\phi(x))$$

And, in general, for any n

$$R_{n+1}(xx'/X) = R_n(xx'/\phi(x))$$

The principle of substitution in its narrower form allows $R_1, R_2, R_3, \ldots$ to be regarded as propositional functions arising from a single function

$$R(n;\ xx'/X)$$

by giving the variable natural number n the values 1, 2, 3, etc. And in all such cases the principle permits the introduction of the function $R(n; xx'/X)$; thus, while (viii) allows the suppression of one of the arguments in a function,

[1] ϕ is, in fact, a descriptive function of the kind defined in *Principia*.

the principle of iteration permits, in special cases, the formation of functions with an additional argument; it does the work, in this system, of the principle which permits the quantification of variable functions in the extended calculus of propositional functions.

The extensions of the principle are (*a*) that it may be simultaneously applied to several propositional functions, e.g. starting from $R(xx'/XY)$, $S(y/XY)$ where x, x', y are variable individuals, Y a variable class of individuals, and X a variable class of couples of individuals, we obtain the classes $\phi(XY)$ and $\psi(XY)$ by (viii), and hence two new functions $R(n\,;xx'/XY)$ and $S(n\,;xx'/XY)$ can be defined by the equations

$$R(1\,;xx'/XY) = R(xx'/XY)$$
$$R(n+1\,;xx'/XY) = R(n\,;xx'/\phi(XY),\psi(XY))$$
$$S(1\,;y/XY) = S(y/XY)$$
$$S(n+1\,;y/XY) = S(\mathrm{n}\,;y/\phi(XY),\psi(XY))$$

(*b*) The class which is substituted is permitted to be a different one at each stage. Such a class would be appropriately symbolized as $\phi(X, n)$ say; the final form of the principle then states that from a function $R(xx'/X)$ we can form a new function $R^*(xx'/Xn)$ by means of the equations

$$R^*(xx'/X1) = R(xx'/X)$$
$$R^*(xx'/X, n+1) = R^*(xx'/\phi(X, n+1), n)$$

With the help of these principles Weyl is able to construct a set of real numbers and a corresponding set of points which possess many of the properties of the Dedekind continuum. These points are 'everywhere dense' on the line, that is every interval of the line, no matter how small, contains infinitely many of them; also Cauchy's principle of convergence is satisfied.[1] This permits the development of the

[1] The form in which the principle is expressed for this purpose is: A sequence $f(n)$ is said to converge if for each fraction α there is some number N, such that for all integers p, $q > N$, $f(p) - f(q)$ lies between $+L$ and $-L$.

theory of differentiation and integration and the introduction of all specific functions such as the trigonometrical, the exponential, etc., used in the early stages of mathematical analysis. It furnishes, however, no support for the more generalized theory of integration developed by Lebesgue and subsequent workers; for the 'continuum' thus defined has many gaps. From the point of view of the pure mathematician who accepts the Dedekind continuum, the points undefined by Weyl's procedure are themselves everywhere dense. In particular the two following theorems do not hold in Weyl's continuum [1]:—

(i) Dedekind continuity: a Dedekind section of Weyl points need not necessarily define a Weyl point.

(ii) Theorem of the upper bound: a bounded set of Weyl real numbers need have neither the upper nor the lower bound.

[1] And most similar theorems containing existential statements concerning real numbers, which are not explicitly defined by converging sequences.

L. Wittgenstein

An (unauthorized) report of some of his views on pure mathematics, which constitute, by implication and explicitly, a thorough repudiation of the logistic thesis.

Dr. Wittgenstein's famous *Tractatus Logico-Philosophicus* has had a profound influence upon the logistic views of the nature of mathematics; in particular the Austrian positivists (the so-called Viennese school[1]) profess to derive their doctrines from him. But the epigrammatic style of that work makes it extremely difficult for the reader to be sure that he has fully understood the important doctrines which are there expounded. The following report is confined to those portions of the *Tractatus* which have direct bearing on the nature of mathematics and, in view of the apology contained in the preceding sentence, must not be regarded as a substitute for first-hand acquaintance with Dr. Wittgenstein's work; the numbers in parentheses always refer to the correspondingly numbered paragraphs in the *Tractatus* from which quotations are made.

Part of the originality of the *Tractatus* derives from its concern with questions of the logical structure of language, i.e. with logical grammar. The answers to these questions in the *Tractatus* are of tremendous importance to any discussion of the nature of mathematics. For it is urged that many confusions in philosophy (and presumably in practical affairs) are due to imperfections in, and misapprehension of, the nature of language; and, further, the indication of how certain specific confusions are to be corrected leads to a

[1] For a general account of this school, see *Die Wissenschaftliche Weltauffassung. Der Wiener Kreis*, 1929.

conception of the integral number and of arithmetic radically different from those of *Principia Mathematica*.

The term *language* in this connection can be used in an extended sense to include any set of symbols used in recurrent combinations for communication between persons; and all such languages are constructed of elements, i.e. any features such as sounds, marks, etc., which can affect the senses and can combine in various ways to form complex symbols.[1]

There are two important aspects of the structure of language to be noted: the presence of structure is partially manifested by the existence of explicitly formulated or implied rules of syntax, permitting the insertion of symbols of a certain kind in any specified context and forbidding the insertion of symbols of any other kind; if such rules are broken nonsense results. It is possible to define *identity of structure* or *equality of multiplicity* between two sentences in terms of the reciprocal possibility of substituting corresponding terms without making nonsense.[1] Differences and equality of multiplicities are manifested by special symbolic devices, *which include the employment of integral numbers as indices*; and this is the second aspect of structure referred to above.

Mistakes made with regard to the multiplicity of sentences can lead to the construction of nonsensical statements; in particular, numbers must not be regarded as elements in the same sense as words, nor must arithmetical equations be confused with ordinary sentences. The basis of the distinction between the two is the very sharp distinction drawn by Dr. Wittgenstein between what can be 'said' i.e. expressed in an ideal language in which all differences of multiplicity are visibly manifested, and what cannot be thus expressed but must be 'shown'. What can be expressed are certain states of affairs or facts, i.e. the existence of configurations of mutually interlocking objects which may or

[1] See p. 24 for a detailed account of the structure of language.

may not occur in the world. A sentence has the same multiplicity as the fact to which it corresponds if the statement it makes is a true one[1]; and what cannot be expressed is what the proposition and the fact (or the proposition and another proposition of equal multiplicity) have in common. If a statement is made asserting the existence of a certain configuration of objects it may be the case either that the configuration does as a matter of fact occur (and the statement is then true) or it may be the case that the configuration in question does not occur (and the statement is then false)—both alternatives are possible; but an arithmetical equation presents no such alternatives, expresses no state of affairs.

If it is granted that mathematical 'statements' are not a species of statement but different in kind, questions as to the status of the former are best answered by returning to the distinction between 'saying' and 'showing'. In the ordinary senses of these words it must appear paradoxical to assert the impossibility of stating what two groups of equal numbers of members have in common; for the obvious answer is to name the common number. This apparent refutation ignores however the difference between two distinct usages of words, namely primarily to refer to objects which are not words and again, in a very necessary subsidiary usage, to refer to symbols themselves; the habitual use of arabic numerals (second usage) as abbreviations for roman numerals or series of strokes (first usage) obscures this distinction. As an illustration we may consider the attempt to express what three groups of four days, four weeks, and four points of the compass have in common; the word *four* used in saying that they are each groups of four can be understood only by knowing that *four* or 4 is an abbreviation for ||||. Thus the symbol *four* functions by drawing attention to the symbol

[1] Multiplicity of *facts* has not been defined, but it should be sufficiently clear how this is to be accomplished, by analogy with the multiplicity of sentences (p. 33).

| | | | which in turn *shows* what there is in common between the three groups; the attempted expression tells us nothing and its function is at most to draw attention to the multiplicity by using symbols in which the latter is shown more obviously. Hence symbols such as | | | | differ completely from symbols such as *red*; it is the meaning of the latter but the shape of the former which is important; and the manner in which a statement such as *A and B have* 4 *members* functions differs entirely from that of statements expressing states of affairs. The conclusion deduced from these arguments is that the natural numbers are indices, i.e. parts of symbols, which serve to make explicit the multiplicity of symbols of which they form part. It should be added however that Dr. Wittgenstein nowhere in the *Tractatus* explains how knowledge is conveyed by 'showing'; nor do other writers, while acknowledging the importance of the distinctions he makes, accept the implication that they preclude the possibility of strict symbolic treatment of the natural numbers in the form of a 'language'. To instance two very different points of view, Carnap [1] regards the exhibition of structure as the only function of language (so that nothing can be 'said') while Chwistek [2] uses the conception of a hierarchy of languages each constituting the subject-matter of the next.

Dr. Wittgenstein is not concerned in the *Tractatus* with the fate of pure mathematics, and though it is clear that his conception of the nature of integers is incompatible with the method pursued in *Principia Mathematica* he does not pursue the analysis of pure mathematics beyond the elementary equations of arithmetic. An account of his analysis of such an equation as $2 + 2 = 4$ will indicate where his theory

[1] "Die physikalische Sprache als Universalsprache," *Erkenntnis*, 1933.
[2] L. Chwistek, W. Hetper, and J. Herzberg, "Les Fondements de la métamathématique rationelle," C. R. M. des séances de la Classe des Sc. Math. et Nat., *Academie Polonaise des Sciences et des Lettres*, December, 1932.

needs supplementation and the difficulties which face such supplementation.

Integers are defined in the *Tractatus* as the indices of an operation—a definition narrower than that given above. An operation is that which must happen to a proposition in order to make another out of it, and so negation, implication, etc., are examples (5.23). An operation is distinguished from a truth function: "the occurrence of an operation does not characterize the sense of a proposition. For an operation does not assert anything; only its result does and this depends on the bases of the operation (operation and function must not be confused with one another)" (5.25)—but this distinction does not seem to be absolutely necessary. The index of an operation is part of the symbol of an operation whose alteration changes the operation into a new one. Thus, if $\sim p$ is written Np and $\sim(\sim p)$ as $N'p$ and $\sim(\sim(\sim p))$ as $N''p$, etc., the strokes in $N''p$ constitute an index. One particular operation is then chosen for the specific definition of integers. Omitting unnecessary complications, the definition (in the present notation) reduces to

$$N^{0}p = p \qquad Df. \qquad (a)$$

$$NN^{x}p = N^{x+1}p \qquad Df. \qquad (b)$$

This is the usual definition by induction and thus contains no novel elements. With regard to this definition it may be noted that all attempt is abandoned to *deduce* the principle of induction as attempted in *Principia*; this is undoubtedly the correct procedure, and all attempts to prove a principle of induction are involved in a vicious circle.[1] But this is a point of view which is not universally accepted; its acceptance entails the rejection of all attempts to deduce arithmetic from logic, for the relations of arithmetical equations to logical tautologies is not that of conclusions to premisses; rather are both to be regarded as exhibiting (from diverse standpoints)

[1] Cf. H. Poincaré (quotation, p. 177).

aspects of the structures of all systems.[1] The view of the nature of pure mathematics inspired by the *Tractatus* may perhaps be not altogether inadequately expressed by saying that pure mathematics is the syntax of all possible systems of symbols.

[1] This view of mathematics is supported by the difficulty of supplying *rigid* proofs of even the most elementary theorems of algebra, e.g. $x \times y = y \times x$. One of the latest such attempts, E. Landau, *Grundlagen der Analysis,* 1929, uses a system of axioms invented by Peano, but the account is written in the usual mathematico-realist manner—induction, and the ideas of existence and collection are used freely. Attempts to symbolize his proofs completely soon prove that a system of algebraic theorems based on arithmetical axioms uses a *non*-formal technique.

L. Chwistek

Another attempt, which should be compared with the section on Weyl above, to obtain a definition of types of functions in terms of the principles used in their construction.

Chwistek's work has been devoted to rebuilding *Principia Mathematica*; this is necessary, first because careful examination of the symbolic conventions used in *Principia* shows their vagueness and in some cases inconsistency, and secondly because as we have seen, the axioms of infinity and reducibility are unmistakably defects in a scheme which purports to contain only propositions belonging to pure logic.

After a very careful and detailed examination of the respects in which *Principia* falls short of symbolic perfection, Chwistek proceeds to elaborate a system whose conventions are stated more explicitly but whose approach to each specific problem agrees fundamentally with those of Russell and Whitehead. The chief novelties are (i) only a finite number of different symbols are used in the system and these are all enumerated, (ii) different kinds of symbols are described by means of a long series of 'directives'. The latter are (non-formal) propositions expressed in words, some of which divide the initial stock of symbols by specific enumeration into a finite number of kinds (which may still be called propositions, individuals, functions, etc.), and others are rules which permit of the construction of new symbols by the use of the logical operations and at the same time state what kinds of symbols are generated by the results of these operations.

The most important of the definitions by induction contained in the directives are the explicit definitions of "being of the same type".

(iii) Ambiguities in the *Principia* definition of the scope of classes are pointed out and corrected; no distinction is made between classes and functions.

(iv) All verbal directions are included in the directives, so that all the proofs are conducted strictly in symbols. This is not possible in *Principia* which needs to use words in some of its proofs.[1]

(v) Metaphysical assumptions are banished from the system wherever possible; thus *individual, function, class, type* are all terms devoid of metaphysical significance and defined merely for convenience of use in a given system of symbols. The individual symbols of Chwistek's system are any simple symbols arbitrarily chosen; and symbols of higher type can be recognized by the fact that they are complex constructs containing symbols of lower type as parts.

(vi) Many of the definitions adopted in *Principia* for their intrinsic philosophic interest can be restricted to serve the special purpose of deducing mathematics from logic. Thus to take one instance, the Leibnizian definition of identity is rejected by Chwistek, for he needs to use identity only between classes. The definition is such that two classes are identical if everything which is a member of one is also a member of the other.

(vii) When quantifiers are applied to variable functions, the functions referred to include the quantifying symbol and must be of a definite type, which is shown by writing one symbol of the same type as a suffix to the quantifier; expressions occur of the type $(\phi)_{\theta(\hat{x})}$ which is read "for all ϕ of the same type as $\theta(\hat{x})$ . . ."

Finally, the axioms of infinity and reducibility are regarded as existential hypotheses which do not belong to pure logic.

[1] Cf. the *Principia* definition of the existential quantifier: $(Ex) f(x) = \sim (x) \sim f(x)$, with Chwistek's $E(x) = \sim (x) \sim$. The former involves understanding the meaning of $f(x)$ (= "any expression containing x"), whereas the latter is a genuine definition in terms of symbols.

So far we have been describing the theory of constructive types. It is essentially the system of *Principia Mathematica* minus all existential propositions, its definitions improved, and its inconsistencies eliminated. No symbols occur or need to be referred to which cannot be obtained by a finite number of operations on a set of initial symbols. This is therefore a kind of logical machine for expressing mathematical theorems in correct symbolism. This system will not suffice to prove more than, e.g., Weyl's can accomplish.

Lately, however, Dr. Chwistek has elaborated a system of semantics,[1] i.e. a symbolic system in which propositions *about* symbols occur and are themselves symbolized. In this system the axiom of infinity is replaced by the possibility of constructing new symbols and it becomes possible to extend the symbolic technique to include the whole of mathematical analysis and the theory of sets of points.[2]

Dr. Chwistek's work is undoubtedly the most thorough attempt to remedy the technical defects of *Principia Mathematica* and the best symbolic system for the logically correct expression of mathematical theorems; as such it has thrown considerable light on the function of such mathematical hypotheses as Zermelo's and the hypothesis of the continuum. In admiring the monumental scale and admirable attention to detail of Dr. Chwistek's work, however, the reader often feels the desire for some discussion of the philosophic implications of his work and its bearing upon the underlying assumptions of the logistic theories. Such an account, which perhaps only Dr. Chwistek himself could furnish, would put his extremely technical discoveries in the philosophic setting their importance undoubtedly deserves.

[1] See f.n., p. 132.

[2] The aleph numbers, however, cannot be defined. Cf. Chwistek: "Une méthode métamathématique d'analyse," *Comptes rendus du Premier Congrès des Mathématiciens des Pays Slaves*, Warsaw, 1929. "Il est sûr qu'il n'y aura pas des alephs, comme il ne peut y avoir des ensembles non dénombrables."

This concludes my account of the logistic theory of *Principia Mathematica* and some of the proposed improvements in that system. It remains only to summarize what appear to be the most important criticisms and to attempt to formulate some conclusions.

Conclusions

The arguments against the logistic system of *Principia Mathematica* are summarized and a scheme of reconstruction outlined.

The reader who has followed the detailed investigation of the logistic thesis in the preceding pages will be in a position to criticize the attitude of mind and method of approach that have inspired our comments. For it will not have escaped his attention that our concern has been primarily with questions of correct symbolization; and indeed the broad generalization, which emerges from a detailed study of the respect in which the logistic programme falls short of accomplishment, is that these imperfections can be traced back to insufficiently precise technique in manipulating systems of symbols. Paradoxical though it may appear to accuse a system as complex and meticulous in construction as *Principia Mathematica* of lack of precision, the preceding sections have shown the inadequacies in the notion of propositional functions, variable, the theory of types, and analysis itself, all of fundamental importance; in fact the elaboration of *Principia Mathematica* is a by-product of the attempt to demonstrate rigorously theorems often verifiable with the help of less complicated symbols, and is not primarily an instrument for analyzing the notions involved.

In view of the knowledge we have now obtained, a critical appraisal cannot be made *inside* the bounds set by a logistic philosophy and, to be complete, would involve far-reaching reconstruction of the general method and Weltanschauung of the system. Such a programme would have both destructive and constructive aspects: the preceding sections have perhaps laid more than sufficient emphasis on the former in pointing out the specific deficiencies of the logistic method as hitherto practised. Such positive suggestions for reform as it has been possible to make have not been organized into a system and the completion of that rather formidable task must be left for another occasion. A constructive attempt of this kind would need to be preceded by a study of the logical structure of language, and take account of the technical researches not only of Russell and Whitehead, valuable as they are, but also of such writers as Peirce (type-token ambiguity, etc.), Wittgenstein (multiplicity, nonsense, etc.), Chwistek (constructive types), Brouwer (reconstructions of theories involving the continuum, etc.), Bernays (Entscheidungsproblem, etc.), Hilbert (axiomatic approach, distinction between sciences and meta-sciences). These writers have provided the foundations of a sophisticated technique for manipulating symbolisms, which would go far to remove the defects of *Principia Mathematica*.

Some indication of the modifications in that work produced by such an approach can be given by formulating in brief the arguments against the propositional calculus. These constitute only a part of the specific defects which would have to be considered in reconstructing *Principia Mathematica*; for the weightiest arguments against that work (and the logistic opinions in general) fall into two classes :—

(i) Objections to the definitions of natural number.

(ii) Objections to the effect that the logistic approach does not clarify the notions of infinity and the continuum.

The criticisms under the first head amount to :—

(*a*) A charge of circularity.

(*b*) A charge of confusing philosophic and systematic logic.

With regard to these it must be said that the *Principia* notion of 'primitive principles' is now quite discredited; for if one is allowed to interpret the marks in the propositional calculus as principles which may be applied to deduce new formulæ, the notions of the independence of axioms, or of the possibility of deducing one formula from another, break down and cannot be clearly defined. It has already been seen that a great many more primitive notions are used in the propositional calculus than are enumerated in *Principia Mathematica*, e.g. rules of significance determining which combinations of symbols are to be significant in the system,[1] and it is impossible to limit the number of non-formal concepts and principles actually used. Again, it was not sufficient for Russell and Whitehead to show that their theorems followed from their axioms, especially if "The method of *Principia Mathematica* is not pursued for the sake of proving $m \times n = n \times m$ but in order to analyse the nature of the entities involved, to exhibit their relations in an orderly manner,"[2] for it has been shown not only that contradictions *may* occur in such systems but that such contradictions do occur in the system discussed and the presence of one contradiction invalidates *all* proofs of the system. Hence Russell and Whitehead should have attempted to show the consistency of the propositional calculus which would in turn have thrown more light on "the nature of the entities involved". This is a demand, however, incompatible with an attitude which believes that selected axioms are obviously true and for that reason cannot lead to contradictions, and its justness is the refutation of that attitude.

These arguments sufficiently prove, in the present writer's opinion, that the *Principia* account of the propositional calculus is unsatisfactory, and that the correct view of such systems requires a sharp distinction (in all subjects) between the *philosophic* and the *systematic* aspects.[3]

[1] These are as important as axioms, for if combinations like $(\sim \vee \sim)$ were allowed, contradictions could easily be deduced.

[2] *Modern Introduction to Logic*, by L. S. Stebbing, p. 177.

[3] Cf. Carnap's two languages, in "Die Physikalische Sprache als Universalsprache der Wissenschaft," *Erkenntnis*, Band ii, Heft 5–6, and also the formalist distinction between mathematics and meta-mathematics (p. 149).

The first of these is non-formal, concentrates upon the subject-matter of which the system (language, branch of knowledge, science) treats, and advances by intuitive insight into a recognition of the nature of phenomena; the latter, presented with a number of symbols, proceeds to assort them systematically, using the symbols themselves as the subject-matter of its investigation. The systematic aspect of a subject is the same as the mathematical deduction of the theorems of that subject.

To these two aspects correspond two distinct uses of symbols, as words with meaning, and as *substitute signs*[1] respectively, words being instruments for thinking about the meanings they express, substitute signs means for not thinking about the meanings they symbolize.[2] If this view is adopted the resulting exposition of the propositional calculus is very different from that of *Principia Mathematica*. No attempt is made to call some formulæ 'primitive', though the relations between the formulæ of the system might be shown, in part, by choosing arbitrarily a set of them for 'axioms' in order to investigate how the rest are then connected with them. In the case of the propositional calculus, however, the mutual interrelationships of the formulæ can be exhibited much more clearly by giving a simple criterion for determining by inspection which formulæ belong to the system and which are excluded from it.[3]

These modifications involve, when pursued consequentially, the surrender of the entire logistic notion of 'deducing' mathematics from logic, but there are compensations. The argument of circularity against the definitions of natural

[1] The terminology derives from Stout, "Thought and Language," *Mind*, 1891.

[2] The matter is, in reality, of course, far more complicated than this account suggests, e.g. thought is always ahead of adequate symbolization.

[3] In technical terminology, the *Entscheidungsproblem* has been completely solved for the propositional calculus. Cf. Hilbert and Ackerman, *Grundzüge der Theoretischen Logik*.

number become harmless, for some kind of formal deduction of the properties of integers is desirable in order to deduce the more complicated theorems concerning them, and it is little more than a matter of convenience whether the natural numbers themselves are taken as given or the propositional calculus is drawn upon for constructs with the same formal properties.

The modifications required to deal with the difficulties associated with the continuum are more radical, for it is with the introduction of the concepts of infinity and the continuum which distinguish the subject-matter of what is conventionally known as pure mathematics from the mathematical method in general, that the logistic system has its most serious breakdown. As we have seen the logistic philosophers are faced with a dilemma between contradictions based on confusion of types or orders and a theory of types which resolves the contradictions only by separating functions into a series of hierarchies which make it impossible to prove many of the results needed in pure mathematics. This is an ancient difficulty and a fundamental one, arising from the fact that a continuum of elements can never be specified by the enumeration of elements, even though that enumeration be indefinitely prolonged.

It has been seen that Dedekind's appeal to common agreement with regard to geometrical intuition is unsatisfactory; but the *Principia* appeal to an axiom of reducibility as a *deus ex machina* is even more so. For Dedekind appealed to common sense to accept the existence of points in specified contexts, and the appeal is at least intelligible; but the axiom of reducibility asserts the existence of *propositional functions*, and the existence of a symbol in the sense required is a notion too vague to appeal to common sense.[1] The

[1] Existence is never defined in *Principia* (the existence of a propositional function must not be confused with that of a class, though both are represented by similar symbols).

solution may be that anything said concerning the *existence* of a function must be interpreted as a metamathematical statement to the effect that the addition of a new symbol to the system will not produce contradiction; but *any* solution must involve the rejection of the naïve conception of propositional functions existing in their own right.

Russell once said,[1] "the very close relationship of logic and mathematics has become obvious to every instructed student. The proof of this identity is, of course, a matter of detail; starting with premisses which would be universally admitted to belong to logic and arriving by deduction at results which obviously belong to mathematics, we find that there is no point at which a sharp line can be drawn with logic to the left and mathematics to the right. If there are still those who do not admit the identity of logic and mathematics we may challenge them to indicate at what point, in the successive definitions and deductions of *Principia Mathematica*, they consider that logic ends and mathematics begins. It will then be obvious that any answer must be quite arbitrary."

We may take up the challenge and reply that the place where the boundary line is to be drawn is outside *Principia Mathematica*. The relation between mathematics and logic is neither identity nor that of conclusions to premisses, but consists in the fact that mathematics must be used in the systematic development of logic (as of all organized systems); and the similarities between logic and mathematics spring from the fact that the first, in its 'philosophic' aspect, is the syntax of possible states of affairs, while the second is the syntax of *all* organized systems.

We conclude that the logistic thesis is not proven, and that elaborate reconstruction can save the technical achievements of the logistic method only at the expense of that method's ambitions.

[1] *Introduction to Mathematical Philosophy*, p. 194.

SECTION II: FORMALISM

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Pure mathematics as the science of the formal structure of symbols.

This section is intended to outline a type of mathematical philosophy usually termed the formalist. The popularity which these opinions have acquired coincides with a general movement in the natural sciences towards greater abstractness of formulation, accompanied certainly by increasing exactitude in the empirical verification of theory but also by apparently increasing unintelligibility of the concepts used. This can be attributed to the use of the mathematical method and a consequent change of attitude among scientists, notably among physicists, towards the objects of their investigations; modern physical theories tend neither to explain the universe nor to describe it, and instead increasingly to exhibit its structure by the use of mutually dependent symbols, unintelligible and meaningless except in specified juxtaposition to other symbols. This transition towards increasing concern with structure, towards increasingly formal character of the concepts used, appears to be connected with the increasing accuracy of the sciences in which it occurs; it is asserted that different observers can agree or disagree only with respect to the *structure* and not with regard to the *content* of their beliefs, and that universality of application and verification of scientific results goes hand in hand with the construction of a formal language to express its results.[1] But it is a mistake to imagine therefore that a scientific system loses meaning in proportion as it becomes formalized;

[1] For a defence of an extremely formalist view of language and science *vide* R. Carnap, "Die physikalische Sprache," *Erkenntnis*, 1932.

its symbols still have 'meaning' in conjunction, in a more complex sense of the word, and it is the attempt to attach associations to individual symbols apart from context that creates the false impression that such systems are mysterious, unintelligible, or meaningless.

If, however, it must be maintained that, however little distinct ideas attach to the individual symbols of physics, the *statements* of physics still have reference to the world of experience, are capable of verification, have meaning, similar considerations applied to pure mathematics seem paradoxically to rob even the theorems of that science of any determinate meaning. This is not to assert that no constant ideas attach to the symbols of pure mathematics, for that would be a manifestly false statement, many mathematical symbols, being older than many symbols of physics, are associated with firmer notions in the minds of those who use them and, in that sense, have more meaning. That is a sense which belongs, however, rather to psychology than to our present considerations and is not intended in the assertion that mathematics has completely indeterminate meaning or reference.

Mathematics, as we have seen (p. 37), may refer to any system of objects and relations whose names can be chosen to ensure that all the initial axioms of pure mathematics are formally true of those objects and those relations. Or expressed otherwise, mathematics is a series of hypothetical deductions from uninterpreted axioms. Thus mathematical theorems have meaning only in an extended use of that ambiguous word; their meaning consists in exhibiting the structure of indeterminate systems; this is the formalist view in brief.

It is a fair criticism of this view to object that its strength lies in what it asserts, its weakness in what it leaves unsaid; but it must be remembered that 'formalism' has always been the working attitude of a group of practising mathematicians rather than a fully explicit philosophy; of interest more for its technical discoveries in the field of symbolism than for the suggestive, but never clearly expounded, philosophy on which it was based.

Formalist views of the nature of mathematics have been powerfully influenced by an evolution towards increasing abstraction exemplified in the history of geometry. Professor Hilbert, the founder of the movement, was responsible for important technical discoveries concerning the interrelation of the theorems and axioms of Euclidean geometry and the possibilities of constructing non-Euclidean geometries,[1] and the technique developed in the course of these researches has profoundly influenced the view of the nature of mathematics held by him and his followers. Formalism is a technique first, and only secondarily a philosophy: a technique for the investigation of the logical interrelation of branches of mathematics and a philosophy to account for the success of that technique. This school has held a particular form of formalist theory with respect to the nature of mathematics in which the whole of mathematics is conceived in the form of theorems, meticulously symbolized, and deduced from (partially) uninterpreted axioms; the validity of these deductions and these axioms being guaranteed by a second science of 'metamathematics', whose subject-matter consists of the symbols of mathematics proper, and whose aim is to demonstrate the self-consistency of mathematics proper with the help of the most elementary and indubitably valid arithmetical methods. If metamathematics can achieve its end, mathematics ensures its own validity, and is interpreted as a formal system of completely indeterminate reference, exhibiting by the multiplicity and interconnection of its own symbols the structure of all possible systems; the set

[1] Cf. D. Hilbert, *Die Grundlagen der Geometrie*.

of mathematical theorems is, as it were, the crystallized syntax of all systems of interrelated objects.

There are two chief objections to this highly ingenious programme; one of principle and one of execution. In principle it errs in neglecting to study the nature and limitations of mathematical symbols themselves; the nature of the initial axioms and the reasons for choice of those axioms rather than others is left completely mysterious. Thus the entire burden of the validity of mathematics is thrown upon the metamathematical proofs of consistency and, presumably, once again upon the mysterious mathematical 'intuition' which dictated the choice of the initial axioms and discovered those parts of mathematics selected for *post hoc* justification. And this leads to the second objection, the fact that it is extremely probable that a metamathematical proof of the consistency of the whole of pure mathematics is impossible. K. Gödel of Vienna seems to have proved[1] that a specific contradiction could be deduced from any proof of the impossibility of the occurrence of contradictions in mathematics. It seems, in fact, that systems like pure mathematics cannot be completely symbolized, and have a multiplicity more complex than any system of symbols which can be devised for the expression of that multiplicity. There is little prospect therefore of ultimate success for the formalist programme in the form advocated by Hilbert and his followers. But in the philosophy of mathematics constructions are not less valuable for being ultimately unsuccessful and it has therefore seemed worth while to supplement the foregoing by an account of—

(1) the development of geometry and science in general towards an increasingly formal aspect;

[1] "Ueber formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme, I," *Monatshafte für Mathematik u. Physik*, vol. xxxviii, 1931.

(2) the Hilbertian view of mathematics;

(3) details of the formalist programme;

(4) a description of Gödel's proof mentioned above.

The last two of these sections are mainly of technical interest.

The Development of Geometry

Documentation of the tendency of a science towards increasing abstraction and concern with structure.

Geometry, at first the practical art of performing calculations required in surveying fields and measuring solid bodies, developed as the study of the actual three-dimensional space in which the Greeks and their followers conceived themselves immersed, and had attained an extraordinary measure of perfection by the time that Euclid wrote his *Elements*. That synthesis of current geometrical knowledge was to remain the textbook of geometers for many centuries. It is a plausible hypotheses that the avidity with which Euclidean geometry was studied by the educated was determined as much by æsthetic as by practical considerations. The Greeks were attracted by the elegance, ingenuity, and clarity of the methods used in proving geometrical theorems, and, from the first, Euclid's *Elements* became a congenial field for logical pedants and connoisseurs of logic; emphasis was always laid upon the necessity for absolute rigour and logical sequence. Nor did this insistence arise from the difficulty of perceiving the truth of the theorems, for Euclidean geometry may be hard to discover but is easy to digest, and the concepts in the *Elements* though, it is true, of a considerable degree of sophistication—'lines' without thickness, 'points' occupying no space—are nevertheless derived from 'actual' lines and 'actual' points in a fashion quite clear at the non-critical common-sense level. Every schoolboy knows how to *imagine* a line as being infinitely thin.

The reasons underlying changes in the character of Euclidean geometry are not to be understood without explicit reference to the traditional plan of presentation—a plan now widely departed from in proportion as its limitations have been recognized by pedagogues.

The *Elements* were arranged as follows: First a number of truths concerning *lines, points, space,* etc., the subject-matter of subsequent theorems. These self-evident truths, to be accepted without proof, are divided into *axioms* and *postulates*.[1] Secondly, the theorems and constructions. In addition there are a number of definitions (e.g. a point as that which has no parts) intended to make clear the nature of the entities mentioned in the proofs, but not all used in demonstrations. The careful distinction between theorems and axioms or postulates is motivated by a desire to deduce the theorems of geometry, i.e. the truths about actual space, by logical deduction alone with no appeal to other sources of knowledge. In particular, diagrams, although in practice indispensable for representing the entities under discussion in the course of long and complicated chains of reasoning, could in theory be entirely dispensed with. Yet the use of diagrams, whether actually drawn or merely visualized, provided a peculiar source of weakness; since they were necessary for facility in demonstrations, no geometer could be certain of avoiding the fallacy of assuming to be necessarily present some feature whose presence was a purely accidental accompaniment of figures drawn or visualized. The crude distinction between 'accidental' and 'necessary' features can be made clearer by an explanation of the manner in which geometrical diagrams are used to promote facility in demonstration.

Diagrams assist the imagination by presenting in succinct

[1] The distinction between *axioms* and *postulates* in Euclid seems to be that the first are self-evident, but the second must be assumed without proof, even although not self-evident. The student who doubts them must become convinced of their truth in the course of the development of geometry, e.g. three of Euclid's five postulates assert the possibility of certain constructions. Cf. Heath, *The Thirteen Books of Euclid's Elements*, pp. 117–151, for a discussion of conflicting views as to the nature of the distinctions.

visible fashion what may prove to be relations whose existence can be independently demonstrated ; and the, at best, crudely inaccurate drawings of black lead on paper or chalk on slate succeed in this function despite their deviations from the ideal ; for although lines drawn on paper cannot be the straight lines of Euclid the geometer can intuit the properties of 'ideal' lines by ignoring such features of the lines actually drawn as their thickness, their deviations from the straight, etc.[1] Generalizing from visible figures involves a trivial danger of mistaking accidental characteristics of the particular figure for essential ones—trivial because hardly likely to be repeated by other geometers—and a serious danger of accepting without proof essential topological facts common to all visible diagrams in the physical space to which diagrams belong.

Euclid fell into the latter mistake.[2] Topological facts are such as are unaltered by continuous deformation of the figure, e.g. the fact that a straight line passing through a point inside a circle will cut the latter when produced. Most geometrical relations except those dealing with lengths [3] (the so-called 'incidence' properties in particular) are topological in the present sense.

Euclid's demonstrations, so long thought to be supreme examples of logical accuracy, contained unproved premisses and even fallacious reasoning. The researches which

[1] Such intuitive isolation of relevant geometrical properties from irrelevant is essential to all deduction. Paradoxically enough, the Greeks thought to do without appeal to a figure, i.e. without intuition, but the formalists show that intuition is essential to correct proof (although *their* diagrams are logical not geometrical ones).

[2] A glaring example is a proposition implicitly used by him again and again : Given two circles, C_1 and C_2, if C_1 passes through a point outside of C_2 and a point inside of C_2, C_1 and C_2 must cut. This seems obvious enough if circles are drawn on paper but neither follows from Euclid's axioms, nor is stated as an axiom.

[3] These topological features of a diagram are essential to the proof, e.g. if it be assumed that the interior bisector of an angle A of a triangle ABC, and the perpendicular bisector of the side BC meet *inside* the triangle (which seems plausible in a rough diagram) it can be easily proved that the triangle ABC is isosceles (which is not true in general). The traditional proofs in Euclid often quietly assume that if two lines meet inside a triangle they must do so.

ultimately led to the discovery of these flaws were not inspired by the expectation of finding them but by the desire for even greater logical elegance; for, among the axioms, the famous parallel axiom (given a straight line, through any point not lying on this line or this line produced a second straight line can be drawn such that however far both lines are produced they never meet) seemed far less 'self-evident' than the rest. It is difficult to 'imagine' a line produced to 'infinity'[1] or, on account of its indefinite character unshared by more 'self-evident' axioms, even to feel convinced of its truth. As it was a constant aim of geometers to reduce the number of proved initial axioms to a minimum it was felt that the parallel axiom might eventually be deduced from the simpler remaining ones. The 'truth' of the parallel axiom, however, was never doubted, and the first who appears to have attempted to deny it was the Italian geometer Girolamo Saccheri (*Euclides ab omni naevo vindicatus*, Mediolani, 1732), and he only as an indirect means of establishing its veracity. Much impressed by the deductive power of *reductio ad absurdum* he conceived the notion of attempting to prove the parallel axiom by deducing a contradiction from the conjunction of the *denial* of the parallel axiom with the other undenied axioms.[2] And so, unwittingly, he proved many theorems in what is now termed non-Euclidean geometry. This is the crux of the whole matter—Euclid's parallel axiom is neither true nor false. For, in the first place, it is now known that the parallel axiom is independent of the other axioms of Euclid, and cannot be deduced from them. And, further, by *denying* it (or modifying it) the addition of

[1] The parallel axiom is now often enunciated in a form in which all mention of infinity is omitted.

[2] Denying the parallel axiom gave Saccheri an extra premiss for his reasonings and therefore justified him in hoping for more success than his predecessors had had. Cf. G. Vailati, "Sur une classe remarquable de raisonnements, etc." *Revue de Métaphysique et de Morale*, 1904, for a good account of Saccheri's work.

the new axiom so obtained to the remaining Euclidean axioms leads to the construction of several self-consistent non-Euclidean 'geometries'. These are the famous hyperbolic and elliptic 'geometries' discovered by Bolyai and Lobatschewsky. The inverted commas round the word *geometries* in the last two sentences emphasize that the word is now being used in a new sense. The Greeks understood the word geometry as the study of actual space, and could not conceive of a plurality of geometries. On such a view the parallel axiom must be either true or false—probably the first, conceivably the second, but certainly one or the other. The embarrassing, but unfortunately valid, possibility of the compatibility of both alternatives destroys the whole basis of this view of geometry. With the discovery of the non-Euclidean geometries and the superfluity of competing geometries, no one geometry could be regarded as a collection of truths about space. And could at best be interpreted as a system of hypotheses of the form *If space obeys the axioms of Euclid then it will have the following properties*: (here would follow the theorems). This view (held by many mathematicians) lays the emphasis on the *deductive connection* between the theorems and the axioms. It is essential in the geometry under consideration that the theorems *follow* from the axioms; it cannot be essential that the axioms should be 'true', for we do not know whether the axioms are true, and shall be able to consider many geometries with different sets of axioms. Now there is a difficulty about the notion of the axioms being true which has been slurred over till this point. For, as mentioned above, the concepts which occur in these axioms are extremely sophisticated, obtained by abstraction from 'real' lines, 'real' circles, etc.; the same process of abstraction is likely to be used again in proofs and to introduce fallacies in the reasoning. Symbols of whatsoever nature are understood only by a process of abstracting relevance from a tangle of irrelevant features, not fundamentally different from the process connected with the use of geometrical diagrams. If the latter could lead to inadequate notions of the nature of geometry the former is also suspect. So two factors converged together to destroy the view of geometry as being a system of hypothetical theorems about space—theorems true of space if the initial axioms are—and to destroy any dogmatic belief in the unconditional validity of geometrical and other mathematical results. First, the desire to know which set of initial axioms was the correct one led to a scrutiny of the nature of the ideal concepts which occur in them; secondly, the desire for accuracy in geometrical proof produced attempts to eliminate possibilities of error caused by the process of intuitive abstraction [1] by which the geometrical concepts were derived.

This question of consistency is of fundamental importance. Some have held that our concepts of space are self-contradictory, others that the truth of axioms about space is synonymous with their mutual consistency, i.e. that there is only one self-consistent geometry and that necessarily the true one. Formal self-inconsistency is disastrous for a geometry, for if a formal contradiction can be deduced, i.e. if two theorems can be proved which contradict one another, then not only those two but any theorems can be both proved and disproved in the system.[2] In such conditions of course the 'geometry' collapses. And the question of the independence of the axioms, which, as we have seen, inspired the earlier geometers, is closely connected with that of consistency. For suppose that the second hypothesis mentioned above was correct and that Euclidean was the only possible geometry; for simplicity

[1] Intuitive abstraction = the process of ignoring irrelevancies, not systematically but by a direct mental act.

[2] This result follows from the truth of the formula $(p \; \& \sim p) \supset q$ in the propositional calculus: a contradiction implies every proposition; therefore if a contradiction can be proved, every proposition can be proved.

imagine that Euclidean geometry is self-consistent and all non-Euclidean ones are self-contradictory. Let P be the parallel axiom and A represent the conjunction or logical product of the remaining Euclidean axioms. Then by hypothesis, *not*-P (the contradictory of P) and A together lead to contradiction, hence P follows from A. In other words, the self-contradiction of all the non-Euclidean geometries implies that the parallel axiom can be deduced from the remaining Euclidean axioms. This suggests why so much of the work of the large school of geometers busy during the last fifty years with the foundations of geometry has been devoted to the proof that geometries were free from contradiction; the fact that no contradictions occur inside a formal system is its most important property.

The investigations of the foundations of geometry have conclusively shown that the non-Euclidean geometries are self-consistent, and have therefore demonstrated that no geometry can be uniquely characterized by the property of being free from contradictions; the last reason for restricting geometry to the study of space has disappeared and the following view of the nature of geometry is generally accepted: a geometry does not deal with space but consists of a series of formulæ (a logistician would say *propositional functions*) which are deduced from a number of initial formulæ (axioms); and any interpretation of the symbols mentioned in the axioms, which converts the latter into true propositions, is an interpretation of the geometry.

With this conception of the nature of geometry there is no reason to distinguish between geometry and algebra or other branches of mathematics, which are all formal systems in the sense indicated. If a distinction is required to be made—as it is in practice since we usually have a special interpretation of geometry in mind—it can be made in some such way as Russell's: "Geometry is the study of series of two or more dimensions" (*Principles of Mathematics*, p. 372), i.e. by restricting the name 'geometry' to abstract systems of particular kinds of complexity, but any such division is arbitrary and conditioned merely by the history and intended applications of pure mathematics.

This completes my account of the development of the science of geometry from the study of space to the study of abstract systems.

What has been said of geometry is true, to a lesser degree, of other sciences, all of which develop in two distinct ways—by rendering the fundamental concepts of the science more precise (e.g. the transition from 'heat' to 'temperature', from 'colour' to 'wavelength') and by discovering and formulating laws of ever-increasing generality. These two processes of growth are interconnected: attempts to classify the fundamental concepts of a subject lead to the discovery of new, and the modification of existing, laws (cf. Einstein's discussion of 'simultaneity'); formulation of new laws promotes the clarification of notions involved in the science, by providing further opportunities for their verification, and may lead to their replacement. Mutual interaction of this kind tends to rob words of their original meaning in return for technical connotations, intelligible only in specified contexts; in extreme cases, the words are regarded as mere instruments for providing numerical results which can be compared with experiment.[1]

The use of the mathematical method, too, often provokes the invention of symbols determined by questions of mathematical exigency and not by the condition of having meaning in isolation. We may suitably conclude this section, therefore, by a very striking example of how mathematical treatment of

[1] "The only object of theoretical physics is to calculate results that can be compared with experiment, and it is quite unnecessary that any satisfying description of the whole course of the phenomena should be given" (P. A. M. Dirac, *Principles of Quantum Mechanics*, p. 7). This may be coupled with Mach's remark, "Science itself, therefore, may be regarded as a minimal problem, consisting of the completest possible presentment of facts with the least possible expenditure of thought" (*Science of Mechanics*, 2nd English edn., p. 490).

physics led to the formulation of new concepts in the history of the discovery of Planck's quantum (see M. Planck, *Origin and Development of the Quantum Theory*, Nobel Prize Address, 1922).

Planck describes how the empirical nature of the simple law connecting the entropy of a resonator and its energy led to the introduction of an absolute value of entropy—"what one measures are only the differences of entropy, and never entropy itself, and consequently one cannot speak, in a definite way, of the absolute entropy of a state. *But nevertheless the introduction of an appropriately defined absolute magnitude of entropy is to be recommended, for the reason that by its help certain general laws can be formulated with great simplicity* "—and to the consequent appearance of an uninterpreted constant—" while this constant was absolutely indispensable to the attainment of a correct expression of entropy . . . it obstinately withstood all attempts at fitting it, in any suitable form, into the frame of the classical theory. So long as it could be regarded as infinitely small, that is to say for large values of energy or long periods of time, all went well; but in the general case a difficulty arose at some point or other, which became the more pronounced the weaker and the more rapid the oscillations. The failure of all attempts to bridge this gap soon placed one before the dilemma: either the quantum of action was only a fictitious magnitude and, therefore, the theoretic deduction from the radiation law was illusory and a mere juggling with formulæ, or there is at bottom of this method of deriving the radiation law some true physical concept "—whose persistent reappearance in many diverse fields led to its incorporation as a fundamental notion—" that the decision [to accept discrete quanta] should come so soon and so unhesitatingly was due not to the examination of the law of distribution of heat radiation . . . *but to the steady progress of the work of those investigators who have applied the concept of the quantum of action to their researches.*"

The Formalist View of Mathematics

Mathematics, if it exhibits structure, does so in complex fashion with the help of 'ideal elements'.

The last two sections have had an apologetic tendency, and must be supplemented by one important criticism if they are not to convey an altogether misleading impression of the plausibility of the formalist doctrines. For to characterize mathematics, as the formalists do, as a science concerned with the exhibition of structure by the employment of symbols meaningless in isolation, is to suggest analogies with the manner in which the structure of concrete systems (families of individuals, portions of a landscape in their physical relationships) can be represented by diagrams (family trees, maps). Such analogies are likely to be misleading in two respects: for the fashion in which collections of mathematical theorems image the structure of the subject-matter to which they may be applied resembles the relation between a landscape and its map only remotely, the arrangement of the former corresponding to the order of discovery of theorems, therefore incomplete, and in process of supplementation. That such supplementation is a necessary feature of any mathematical symbolism is a consequence of the fact that mathematics treats of infinite systems. Any view based upon a strict symbolizing of mathematics, as is the formalist, will have to admit among its symbols some incapable of interpretation either in isolation or in specified contexts. Thus the comparison between the present state of physics and the formalist view of mathematics must not be pressed too far; the latter is characterized by the presence of the so-called 'ideal elements', meaningless symbols (described in greater detail below) which can never appear in any final theoretical formulation of physical truths.

Thus, in so far as formalist mathematics contains unexplained 'ideal elements', it will require further explanation; and the (probable) impossibility of completing the formalist programme makes such a justification imperative.

So far the easiest explanation has been no explanation, i.e. the ideal elements have been explained as purely symbolic devices [1]; but freedom to ignore their interpretations is limited by the necessity to justify their introduction by proof of consistency and vanishes when it is found impossible to produce the latter.

Finally we may sum up the formalist view of mathematics as follows: the typical mathematical method is the investigation of structures of systems by the use of systems of symbols of indeterminate reference, arranged in the form of theorems deduced from axioms, and containing 'ideal elements'; the employment of the latter is essential, and must be legitimized by proofs of consistency.

The Formalist Programme in Detail

A technical summary, for specialists, of the axioms and symbolic innovations of the formalist school.

The programme aims at proving successively that one branch of mathematics after another is free from contradictions. This is to be accomplished by symbolizing mathematics and

[1] An 'ideal', I, is a symbol whose addition to a system of formulas, with appropriate modification of the axioms, extends S in such a way that the new system, S' say, agrees with S in respect of *all* formulæ not involving I. That a symbol I is an 'ideal' with respect to a system S requires proof. 'Ideals' function by exhibiting the structure of systems S as partial sections of (often simpler, more uniform) systems S'. E.g. 'the point at infinity' introduced into Euclidean geometry exhibits the relationship between the latter and projective geometry.

logic simultaneously, i.e. by constructing a formal system containing symbols for mathematical functions, numbers, etc., as well as logical constants, propositions, etc. There are various novelties of notation which will be described as they arise. The system[1] begins with a propositional calculus employing the usual signs and the following axioms :—

I. *Axioms of Implication*

1.1. $p \supset (q \supset p)$
1.2. $(p \supset (p \supset q)) \supset (p \supset q)$
1.3. $(p \supset (q \supset r)) \supset (q \supset (p \supset r))$
1.4. $(p \supset q) \supset ((r \supset p) \supset (r \supset q))$

II. *Axioms of* or *and* and

2.1. $p \,\&\, q \supset p$
2.2. $p \,\&\, q \supset q$
2.3. $p \supset (q \supset p \,\&\, q)$

2.4. $p \supset p \vee q$
2.5. $q \supset p \vee q$
2.6. $((p \supset r) \,\&\, (q \supset r)) \supset ((p \vee q) \supset r)$

Note : & = *and* ; v = *or* ; &, v bind more tightly than ⊃.

III. *Axioms of Negation*

Principle of *Reductio ad Absurdum*, viz.,

3.1. $(p \supset q \,\&\, \sim q) \supset \sim p$

Axiom of double negation, viz.,

3.2. $\sim \sim p \supset p$

In addition two rules of manipulation are used, viz. those of substitution and the syllogism.

Note : The increase in number of these axioms as compared with those used in the logistic calculus of propositions is due to the change in the purpose for which the axioms are to be

[1] i.e. the system used by Hilbert (see " Die Grundlagen der Mathematik ", *Abh. des Math. Seminars zu Hamburg*, vol. vi, 1928).

used; the axioms now under consideration were chosen principally in order to simplify proofs of consistency, the question of independence being subsidiary.

IV. *The logical ϵ axiom*

$$4.1.\quad A(x) \supset A(\epsilon A)$$

The ϵ notation was invented by Hilbert in order to eliminate use of the quantifiers. If Fx is any propositional function, ϵF may be interpreted as denoting any individual, say a, which is such that $F(a)$ is certainly true if there is *some* x for which Fx is true. The following formulæ allow $(x)Fx$ and $(Ex)Fx$ to be defined in terms of the ϵ notation.

$$(x)Fx \equiv F(\epsilon \sim F) \text{ and } (Ex)Fx \equiv F(\epsilon F)$$

e.g. let $Fx = x$ *is corruptible*; ϵF can then be interpreted as denoting the most corruptible man (or nobody if nobody can be bribed). Here if *somebody* can be bribed we know that ϵF can certainly be bribed.

V. *Axioms of Equality*

5.1. $(Za) \supset (a = a)$

5.2. $(Za \,\&\, Zb) \supset ((a = b) \supset (A(a) \supset A(b)))$

Za means a is an integer (Z for Zahl = number).

VI. *Axioms of Number*

6.1. $(Zx) \supset (x' \neq o)$

Principle of mathematical induction, viz.,

6.2. $(Za) \supset [\{A(o) \,\&\, (x)(A(x) \supset A(x'))\} \supset A(a)]$

In addition to the above, the so-called 'primitive numbers' are used, viz. the signs 0, 0′, 0″, etc.

In order to restrict the ranges of the variables occurring and to distinguish between these variables, Hilbert adopts the device exemplified in the above four axioms of preceding all expressions in which the variable occurs by the sign of implication and an expression typifying the variable. Thus

Za means a is a natural number. Every variable that occurs will be associated in this way with a typical function characterizing it which must appear in all expressions in which the variable appears, just as Za, Zb appear in the axioms of the last two groups above. For example, a variable f will be needed whose range is that of those functions of integers whose values are integers. This has the characteristic function $\phi(f)$ which is an abbreviation for $(x)[Zx \supset Z(fx)]$. These conventions are very convenient as they obviate the necessity for making those distinctions between different types of variables which produce such complexity in most formal systems.

The above completes the list of general axioms required. The various mathematical operations and functions can now be introduced into the scheme either (*a*) by explicit definition or (*b*) by induction. In (*a*) a formula is given which allows the sign for the function in question to be eliminated in one step from any expression in which it occurs, e.g. $Zx \supset (fx = x)$ would be a definition of the function f of integers which always has the same value as its argument. In (*b*) formulæ are given which allow the sign for the function to be eliminated in a finite number of steps *whenever primitive numbers are* substituted for all the variables, e.g. definition of addition of integers :—

$$0 + 0 = 0 \quad (1)$$

$$0 + 0' = 0' \quad (2)$$

$$0' + 0 = 0' \quad (3)$$

$$(Zx, Zy) \supset (x' + y = (x + y)') \quad (4)$$

$$(Zx, Zy) \supset (x + y' = (x + y)') \quad (5)$$

In both (*a*) and (*b*) the formulæ in question are added to the preceding axioms and treated as new axioms. In case (*b*) this procedure is permitted only when it can be *seen* that the definition satisfies the condition mentioned above, viz. that any formula in which no variables occur, e.g. $0''' + 0''$ in the case considered can actually be reduced to a primitive number. How this is done in this particular case may be instructive.

We have $Z0'''$ and $Z0''$. Also $0''' = (0'')'$
$\therefore$ by substitution in (4) $0''' + 0'' = (0'')' + 0'' = (0'' + 0'')'$.
And similarly $0'' = (0')' \therefore 0'' + 0'' = (0')' + 0'' = (0' + 0'')'$.
And again $0' = (0)' \therefore 0' + 0'' = (0 + 0'')'$

Now using (5) in the same way we get $0 + 0'' = 0 + (0')' = (0 + 0')'$. But $0 + 0' = 0'$ by (2) $\therefore$ by successive substitution $(0''' + 0'') = ((((0')')')')' = (((0'')')')' = ((0''')')' = (0'''')' = 0'''''$.

The ϵ operator defined above is used to obtain the so-called transfinite mathematical functions whose values, though theoretically determinate, can only be found in exceptional cases as their determination involves the performance of infinitely many operations (e.g. the function $f(n)$ of integers which takes the value 0 or 1 according as $n^{\sqrt{n}}$ is rational or irrational).

The effect of ϵF when F is a propositional function is to choose a value of the argument of F which makes Fx true. Similarly, when f is a mathematical function of integers, $\epsilon' f$ can be interpreted as follows: if fx is 0 for all x, $\epsilon' f = 0$; otherwise $\epsilon' f$ denotes the least integer for which $fx \neq 0$. Clearly $\epsilon' f$ as so defined is a transfinite function of f for, in general, there may be no way of finding the least integer. It can easily be seen that $\epsilon' f$ is equivalent to $\epsilon(f = 0)$.

We can now proceed to define the real numbers by means of dyadic decimals, i.e. as a function $\phi(x)$ of integers whose only values are 0, 1. ϕ will, of course, need a characteristic formula asserting this. It is:—

$(x)(Zx.Zy) \supset ((x)\ (\phi x = 0 \vee \phi x = 1).(x)\ (Ey)\ (\phi\ (x + y) = 1))$

Let us call this expression $R\phi$. A sequence of real numbers can be defined by means of a function $\phi(x, y)$, where $Zx . Zy$ is true, and such that $(y)R\phi(x, y)$.

Similarly all the functions which occur in the theory of real functions can be defined. The fact that these definitions involve no more than the notions defined above shows clearly that the difficulties associated with the transfinite definitions, are exactly equivalent to those produced by the use of the ϵ notation.

Note on Gödel's Theorem [1]

A mention of the remarkable theorem which purports to demonstrate the impossibility of proving mathematics to be free from contradictions.

Gödel demonstrates that a specified class of systems, including a restricted [2] calculus of propositional functions substantially agreeing with that of *Principia Mathematica*, is characterized by the peculiar fact that each such system will contain theorems which can be seen to be true but do not permit of formal demonstration according to the rules of the system. One such theorem is described, and it is shown that the formal demonstration of this (true) theorem in the calculus of propositional functions would lead to a contradiction. Thus that calculus, and many similar systems, are incomplete in the sense that some of the true theorems concerning the subject-matter of their axioms are incapable of formal deductive demonstration in the systems.

This remarkable result is obtained as the climax of a mathematical proof, involving forty-six cumulative definitions, and therefore, perhaps, too complicated to be described in

[1] K. Gödel, "Ueber formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme, I"; *Monatshefte für Mathematik und Physik*, xxxviii, 1931.

[2] i.e. with quantification restricted to the arguments of propositional functions and not applicable to functions themselves.

this place; but the lines of the proof can be indicated. It is based on the ingenious notion of replacing the brackets, logical constants, and all other signs in the propositional calculus by numerals, a transformation which is, of course, perfectly permissible. All formulæ and, in particular, the demonstrable theorems of the system are thereby transformed into sequences of integers. Further, statements concerning these formulæ (e.g. such and such a formula follows from the initial axioms) can be expressed in the symbolism of the propositional calculus and so, ultimately, transformed into a sequence of integers. The chain of definitions, referred to above, performs this process in detail and is used to produce a formula whose formal proof is shown to be impossible.

In a later section Gödel proves that a contradiction could be deduced from any proof that the *entire* calculus of propositional functions could be formalized in the same fashion as the restricted calculus above. This is a very important result for, if correct, it seems that the calculus of propositional functions will not permit of the complete symbolizing required by formalist proofs of consistency. The reader must be referred to Gödel's paper for further details.[1]

[1] Cf. also P. Bernay's report, "Methoden des Nachweises von Widerspruchsfreiheit und ihre Grenze," at the International Congress of Mathematicians, Zurich, 1932.

SECTION III: INTUITIONISM

Static and dynamic attitudes to pure mathematics.

THE progress of mathematics is not smooth, nor is the science, as the layman imagines, a collection of subtle principles and infallible results, springing mysteriously yet convincingly into the minds of their inventors. Its discoveries have, in general, not won immediate or universal acceptance, for mathematics, like every other system of organized knowledge, owes its development to the insight of thinkers whose creative imagination has led them to results which often startled themselves and their contemporaries; it is the crystallization of an activity more certain of its results than its principles. Yet, a result once generally accepted by mathematicians is seldom retracted, and then only with great pangs; for this science has a certainty unchallenged by any other department of human knowledge. Its practitioners willingly conceive of it as an unchangeable system of eternal truths, an interrelated system of theorems which may be extended but not controverted. This type of attitude is essentially static; conceiving of a science as if it were a library, which acquires new volumes but never destroys the old, and therefore obviously inappropriate to sciences like physics, where violent revolutions are still the order of the day, it exercises a great deal of influence on philosophies of mathematics, on account of this distinguishing element of certainty in mathematical theorems which is so hard to explain.

Nevertheless, this static attitude towards mathematics demands an ideal science which always advances and never makes mistakes. When it is held by mathematicians, at any particular stage in the history of mathematics, it is an expression of their hopes rather than of their convictions.

In the static vision mathematics is regarded as a body of truths whose certainty is unchallengeable. Yet these same truths are the product of an historical process of development, in the course of which principles have been freely employed and theorems accepted as true which were later seen to be false. So there is good reason to believe that some at least of the truths and principles now regarded as eternally true will be rejected by future generations of mathematicians. Hence the supporter of the static vision, in spite of himself, is inevitably driven to defend his position by arguments which will display the principles of growth of his science, or at least ensure that the theorems he postulates true will not be controverted.

The philosophies which have been most influenced by a static attitude have been the formalist and the logistic, and we have seen how their supporters have attempted to justify their opinions, the former by reducing mathematics to logic, the second by proofs of consistency. Strictly speaking neither of these methods results in a principle, but, if correct, either would ensure the validity of mathematics and make the static view a possible one, the assembly of mathematical truths preserved in the first case being those which can be strictly deduced from the primitive axioms and in the second case those which had been safeguarded by proofs of consistency. Both of these philosophies are dogmatic ; they are *a posteriori* justifications of a static attitude towards mathematics, and suffer from the usual vulnerability of all dogmas in needing to be invulnerable. Refutation in a single instance destroys the infallible.

Let us now examine an alternative attitude towards mathematics which may be called the dynamic, for want of a better word. This is a type of attitude in which emphasis is on the growth of the science, rather than on its invulnerability. Mathematics is now regarded above all as a product of the activity of fallible human minds and, as such, liable to be affected by all the defects to which our thought is essentially subject. Thus just as the supporters of static attitudes will tend to emphasize the external forms of mathematics, its formulæ *qua* physical objects, just because these are the most permanent and tangible features of mathematical activity, so also the supporters of dynamic evolutionary attitudes emphasize mathematical *thought* just because it is that element in mathematics which is most intangible, changing, and capable of development. The dynamic attitude is consistent with an evolutionary conception of history and naturally arises from it, since a general progressive movement in history will account for the certainty of mathematics, which is seen now as a progressive and approximatory tendency, a process rather than a characteristic.

The two types of attitude I have sketched occur together in the minds of most philosophers of mathematics, and the ways in which the ensuing tension is resolved is characteristic for each philosophy. According as the static or the dynamic side of the opposition is given preference, different problems have to be faced by the resulting philosophy. Thus predominantly static philosophies of mathematics have to account for the development of the science, and to explain the possibility of error, etc., while the predominantly dynamic philosophies will be called on to face the awkward problems of the 'universality' and 'certainty' of mathematics.

For the logisticians part of the problem is to explain mathematical discovery; if, as they say, mathematical theorems are obtained from logical tautologies by means of logical deduction, how is any advance in mathematical knowledge possible? For in one sense logical deductions add nothing to knowledge since all that is contained in the conclusion was already contained in the premisses. The answer that the logistician must give is to distinguish between knowledge and the discovery of that knowledge. Mathematical discovery takes place by a process of trial and error. Having chosen a formula which for some reason or other the mathematician believes may be true, he experiments with various true premisses until he finds a combination from which he can either prove or contradict his theorem. And this process is needed just because there is no uniform method for proving all true formulæ in the calculus of propositional functions. We can even see why this is the case. When, in the course of a deduction, the syllogistic principle is used to deduce B from the two statements A and $A \supset B$, the symbol for the conclusion is already contained in those of the premisses. In the converse process, however, B is given and we must look for such an A that $A \supset B$ is a theorem already proved, i.e. in the process of mathematical discovery there is an element of synthesis. In order to prove B we must first synthesize the formula $A \supset B$.

Those who reject this solution, however, and believe that mathematics cannot be deduced from logic will have to allow some typically mathematical mode of knowledge, some principle which is characteristic of mathematics. Now we shall find the opinions of the philosophers who have been classed together as intuitionists all agree to the extent that they assert that mathematics is based upon a fundamental intuition of some process or principle which is not capable of deduction from tautologies and is therefore synthetic in character. And we shall find that most of them agree in emphasizing mathematical thought and distrusting the excessive use of symbolism.

I may now sum up this introduction. There are two possible aspects of mathematics, the static and the dynamic, and according as the one or the other is specially emphasized we get sharply contrasted types of philosophy of a predominantly static or dynamic character respectively. The former is sympathetic for philosophies with realist tendencies, and the latter for idealist. The intuitionists are inspired by the second type of attitude.

The Mathematical Predecessors of the Intuitionists

Some account of the opinions of Kronecker and other early intuitionists, with a digression on the theory of sets of points.

This section is devoted to an account of opinions on questions of mathematical philosophy which were held by certain eminent mathematicians during the years which immediately preceded the full development of intuitionism. It was a period which saw, on the one hand, the arithmetization of mathematics accomplished as a result of the brilliant researches of Weierstrass, and, on the other hand, the development by Cantor of the theory of transfinite numbers and the modern theory of sets of points. The work of Weierstrass, with his brilliant contemporaries and successors, gave the pure mathematician an extremely powerful analytic apparatus for handling questions in the theory of functions. Their discoveries revealed the imperfections and fallacies involved in the work of the pure mathematicians who had immediately preceded them and set a new standard of accuracy. The work of this period is characterized by a continual tendency towards abstraction and generality. Once it was realized that the old concept of function concealed surprising subtleties the way was clear for an extremely general conception of function which in turn led to generalizations of such notions as integration, convergence of series, etc. This tendency was encouraged by the success of Cantor's theories, which not only appeared to tame the infinite once and for all, making it amenable to mathematical treatment, but revealed a veritable mathematical paradise of infinities upon infinities, each with its own cardinal number to fit into a correct place in an unending hierarchy. Its properties proved capable of immediate application to the growing theory of functions, where it allowed the subtlest distinctions to be made concisely and accurately.

Yet, for all this success for the transfinite method, throughout this period, even before the discovery of the contradictions in the Cantor theory of cardinal numbers, some of the greatest mathematicians protested against the prevailing tendency and tried to persuade their contemporaries, though with little success, to renounce their methods. Much of their opposition may no doubt be ascribed to the inevitable reaction produced by any victorious movement, but nevertheless an examination of the opinions advanced by these reactionaries demonstrates how the problems, which the modern intuitionist claims to have solved, arise inside the very body of mathematics and exercised from the very first the minds of some of those who contributed most to its development in modern times.

The most striking of these early forerunners of Brouwer is perhaps the algebrist, Kronecker (1823–1891), who was a colleague of Weierstrass at the University of Berlin and a very famous mathematician. Weierstrass had tried to demonstrate that all mathematical entities could be developed as constructions of natural numbers; Kronecker went farther and declared that only the natural numbers were 'real', and that *all* mathematical results must actually be results about the natural numbers. Thus not only were irrational numbers, fractions, and complex numbers never to occur in mathematics, but even negative numbers were taboo. As Kronecker himself said in a striking sentence, which will perhaps bear repetition once more, "God made the natural numbers; all the rest is man's handiwork." He appeared to believe that the extensions of the number concept were due to the application of pure mathematics to the physical sciences, and said "I also believe that we shall succeed some day in arithmetizing the total context of all these mathematical disciplines, [i.e. analysis and algebra], that is in grounding them on the number concept taken in its narrowest sense, and thus eliminate the modifications and extensions of this concept which were for the most part occasioned by applications in geometry and mechanics" (Kronecker, "Ueber den Zahlbegriff," *Journ. für Reine u. Angewandte Math.*, ci, 1887, p. 338). The method adopted to rid mathematics of these illegitimate numbers was to replace all equations in which they occurred by appropriate algebraical congruences. An example will illustrate this better than a description. The equation $7 - 9 = 3 - 5$ is illegitimate according to Kronecker on the ground that the expressions on either side denote nothing, the number -2 having no existence. The equation must therefore be transformed into the congruence

$$7 + 9x \equiv 3 + 5x \text{ (modulo } x + 1)$$

(See Kronecker, ibid., p. 337). In this manner the resulting expressions obey the same laws of combination as the original equations, as may be easily verified, so that formally it is possible to manipulate the congruences in the same way that equations involving integers would be manipulated. Difficulties however arise when the congruences which contain a free variable x, and therefore have no determinate meaning, have to be determined in such a way as to reproduce results expressed determinately by the original equations between integers. This Kronecker does by putting $x + 1$ equal to 0, for a congruence modulo zero becomes an equation, and an exact correspondence is obtained between the congruences so obtained and the original equations. In Kronecker's own words, "The congruence transforms directly into the equation as soon as x is regarded no longer as a variable but as a 'magnitude' defined by $x + 1 = 0$ and thus introduces the 'negative unity' (loc. cit, p. 345). The last step of equating $x + 1$ to 0 is however illegimate for this could only be possible if x could take a negative value which by hypothesis is not allowed. Hence Kronecker's constructions for eliminating negative numbers beg the question. Similar arguments apply to his attempts to eliminate fractions by means of congruences modulo several simultaneous bases, and complex numbers by congruences modulo $1 + x^2$. His transformations, besides being logically unsound, completely obscure the relations between rational and the irrational numbers towards which they converge, to mention one example out of many possible ones, and thus are completely impracticable.[1]

Another figure who eminently deserves attention is Henri Poincaré whose outstanding mathematical achievements earned for him a great reputation among mathematicians, while the vigorous and witty style of his more popular writings gave him the ear of a very extensive public. He consistently attacked the logisticians and the formalists although himself a formalist in his attitude towards geometry. His arguments against them, when disentangled from their polemical setting, amount to the charge of circularity. It is interesting to note, however, that he charges the formalists also with circularity, maintaining that they base arithmetic and, eventually, the rest of mathematics on axioms which include an axiom of induction. Yet, in the proofs of consistency which alone justify them in using these axioms, they are compelled to prove results for *all* possible proofs, i.e. for formulæ which may contain any number of symbols.

"Then in order to establish that the postulates do not involve contradiction, we must picture all the propositions that can be deduced from these postulates considered as

[1] Cf. Couturat, *De l'Infini Mathematique*, pp. 603–616, for a full discussion and criticism of Kronecker's views.

premisses and show that among these propositions there are no two of which one is the contradiction of the other. . . . If the number of the propositions is infinite . . . we must then have recourse to processes of demonstration, in which we shall generally be forced to invoke the very principle of complete induction that we are attempting to verify " (*Science and Method*, English translation, p. 152).

This argument seemed to spring from an incomplete understanding of the formalist method but, nevertheless, deserves attention, for it shows clearly the necessity for the use of a non-formal principle in the foundations of mathematics.

Poincaré, by asserting that the integers were indefinable and that the whole of mathematics is based on the principle of mathematical induction whose validity must be intuitively recognized, adopted an intuitionist position in effect, and clearly enunciated doctrines which are still basic parts of the intuitionist philosophy.

The remaining mathematicians to be considered in this section form a group consisting of Borel, Baire, and Lebesgue, sometimes called the Paris School of pure mathematicians, together with Hadamard, whose position conflicted with that of the other three. These eminent mathematicians expressed their opinions in letters to one another,[1] occasioned by the publication of Zermelo's proof that every set could be well ordered. In order to explain how the controversy arose a somewhat lengthy digression into the elementary theory of sets of points will be necessary. The reader who is familiar with this subject may omit this section.

Digression on the Theory of Sets of Points

The purpose of this section is to sketch the theory of sets of points, or rather of the theory of cardinal numbers which

[1] Subsequently published as " Cinq lettres sur la théorie des ensembles " : *Bulletin de la Societé Mathématique de France*, 1905.

forms part of it, sufficiently far to make the Borel-Baire-Lebesque-Hadamard discussion about Zermelo's axiom of choice intelligible to the reader with no previous knowledge of the subject. The philosophic difficulties which arise will be the familiar ones of the conditions in which mathematical entities (in this case sets of points) can be said to exist.

We commence with the idea of *sets* or *classes* of objects. For present purposes this is taken as an undefined primitive idea. It is convenient to talk as if a class or set were a collection of its members in the usual mathematico-realist manner, even when a class has an infinite number of members. Let capital letters A, B, C, etc. be used to denote classes, and let typical members of such classes be denoted by the corresponding small letters a, b, c, etc. The latter can be further distinguished by suffixes when necessary. Thus, in general, a will be a member of the class A. The fact that a thing, x say, belongs to a class Y say, is denoted by $x \epsilon Y$. The class whose sole member is a is denoted by (a).

We must now define the *addition*, *product*, and *similarity* of classes. The *sum-class*, ΣA, of a number of classes A, is the class consisting of all those things which are members of one or more of the A's. This sum-class is supposed to exist whether the number of A's is finite or infinite. If there are a finite number of A's, say $A_1, A_2, \ldots A_k$, whose sum-class is being constructed, it is denoted by $A_1 + A_2 + \ldots + A_k$.

The *product-class*, ΠA, of a number of classes A, is the class consisting of those things which belong to every A of the set considered. Here again the number of A's may be finite or infinite. If the former, and the A's considered are denoted by $A_1, A_2, \ldots A_k$ say, the product class is denoted by $A_1 A_2 \ldots A_k$.

Two classes A, B are said to be *similar*, written $A \sim B$, if there is a one-one correspondence between them, i.e. a correspondence in which each member a of A has exactly one member b of B corresponding to it, and *vice versa*. Similarity is a transitive and symmetrical relation, i.e. if $A \sim B$ and $B \sim C$ then $A \sim C$, and if $A \sim B$ then $B \sim A$. Two classes which are similar are said to have the *same cardinal number* or the same *power*. It follows at once from the fact that *similarity* between classes is a transitive and symmetrical relation that, if some particular cardinal number is defined as the cardinal number of some definite class, A say, the *same* cardinal number is obtained when any class similar to A is substituted in the definition.

Having defined these cardinal numbers we have, if possible, (i) to define ways of combining them analogous to addition, multiplication, exponentiation, etc., of ordinary integers, and (ii) to see how the distinction between finite and infinite cardinal numbers is made.

First of all we notice that, given a pair of classes A, B, it is always possible to construct a pair of *exclusive* classes A', B', with $A \sim A'$ and $B \sim B'$. For let x, y be distinct entities. If A' is taken to be the class of ordered pairs $a' = (a, x)$ and B' the class of ordered pairs $b' = (b, y)$; the one-one correspondence between the members of A and A' is that by which each a' is made to correspond to the a which occurs in it, and similarly for the B' and B. A', B' are exclusive; none of the members of A' can coincide with the members of B', since each of the former is an ordered pair of things of which the second is x, and each of the latter is an ordered pair of things of which the second is y, and x is not the same entity as y.

Now to define the addition of two cardinal numbers: Let A, B be any two exclusive classes of cardinal number α, β respectively. Then the *sum* of α and β, written $\alpha + \beta$, is defined as the cardinal number of $A + B$. In order to see that this definition does not depend on the choice of the particular exclusive classes of powers α, β respectively, it is sufficient to see that if A, B; A', B' be exclusive pairs of classes with

$A \sim A'$ and $B \sim B'$ then $A' + B' \sim A + B$. This is so because we can set up a one-one correspondence between the two classes $A' + B'$, $A + B$ in which a member of the first, if also a member of A', corresponds to the member of A (which is also a member of $A + B$) to which it corresponded in the correspondence by virtue of which $A' \sim A$, and, if a member of B, corresponds to the member of B which was its partner in the correspondence which made $B' \sim B$. This is hard to say but easy to see.

Similar definitions are given for multiplication and exponentiation of cardinal numbers. The method in each case is to take any two particular sets A, B of powers α, β (if it is a function of two cardinals that is being defined) and to define the required function of α and β as the cardinal number of a new set constructed out of A and B by a definite procedure. And each such definition requires a proof that which particular A and B are chosen is irrelevant provided they have the cardinal numbers α and β respectively.

Thus $\alpha \times \beta$ is defined as the power of the class of all the ordered couples (a, b) when a is any member of A and b any member of B.

The *null-class* is defined as the class which has no members, i.e. the class Λ such that $x \epsilon \Lambda$ is false for all x; and the *unit-class* as the class which contains some term x and is such that, if y is a member of it, $y = x$.

It is now natural to define the relations of "greater than" and "less than" between cardinal numbers. This is done as follows:—$\alpha > \beta$ (or $\beta > \alpha$) if and only if it is true that (part of A) $\sim B$ but it is not true that $A \sim B$. Now by analogy with the properties of natural numbers it may be presumed that $\alpha > \beta$ is incompatible with $\alpha < \beta$, i.e. that we cannot have (part of A) $\sim B$ and (part of B) $\sim A$ unless $A \sim B$, but the proof of this depends on the so-called Schroeder-Bernstein theorem.

Let us now take breath for a moment and consider the structure that has been erected. We have been referring quite uncritically to classes and entities 'existing', two classes were said to be similar "when there *is* a one-one correspondence between them" and in the last paragraph we spoke even of (part of A) being similar to another set. If *class* is taken to mean what the logisticians mean by the term, the development of the theory of cardinal numbers as given above becomes identical with the logisticians' development of the theory of cardinal numbers. Hence the question as to *when* a similarity can be said to exist between two classes is exactly equivalent to the old question as to when propositional functions can be said to exist. If A, B are classes with a finite number of members each it can easily be tested whether a one-one correspondence between them can exist; for it is sufficient to look at each of the finite number of possible correspondences in which each member of the one has one or more partners in the other in order to discover whether any of these correspondences are one-one. In the case of sets with infinitely many members this procedure is inadmissible, and it is these sets for which the problem is acute.

Let us consider a concrete difficulty: $\aleph_0$ is defined as the cardinal number of the class of the finite cardinal numbers 1, 2, 3, . . . Now if a class is known to be infinite it would seem natural to suppose it must contain at least $\aleph_0$ terms. There are infinite classes, according to the Cantor theory, which we have been describing, which contain *more* than $\aleph_0$ points, i.e. which cannot be put into one-one correspondence with a class of power $\aleph_0$. The set of all real numbers between any two given numbers will serve as an example. It seems at first obvious that such a set must contain a subset of $\aleph_0$ members, but 'seeing' is not the same as 'proving'. How could it be proved that an infinite class A contains $\aleph_0$ terms? The kind of schema which is behind the intuitive belief that A contains at least $\aleph_0$ members is somewhat as follows:—

A has a member; select one, x.

$A - (x_1)$, i.e. the set of all members of A except x_1, has a member; select one, x_2, etc.

This process can never come to an end, for else the set would have only a finite number of members, hence the set must contain $\aleph_0$ members, viz. $x_1, x_2, x_3, \ldots$

The difficulty about this argument is that, in general, no method can be given for making the choices. The $\aleph_0$ terms have to be chosen by an infinity of successive choices, each of which is dependent on the previous ones, since it is restricted to those members which have not been previously selected. Can the set of members so chosen be said to exist? If this can be assumed many striking theorems can be proved; but without this assumption the whole system remains very incomplete.

This was, very crudely, the problem Zermelo had to face when he tried to prove that every set could be well-ordered (see Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," *Mathematische Annalen*, vol. lix, pp. 514-16). He was the first to use explicitly an axiom which allowed the infinite acts of choice we have mentioned.

The axiom may be put in several equivalent forms. In the following form: "Given any class of mutually exclusive classes, of which none is null, there is at least one class which has exactly one term in common with each of the given classes" it is often called the multiplicative axiom, since it has to be used in defining the product of an infinite number of cardinal numbers.

The Mathematical Controversy

We may now return to the controversy between the French mathematicians. Borel inserted a short note into the *Mathematische Annalen*, criticizing Zermelo's use of the multiplicative axiom, and thus provoked a reply from Hadamard in the first of the 'cinq lettres'.

The latter distinguishes between the existence of mathematical correspondences and their description, and asserts that correspondences or functions may very well exist even although we have no way of describing them. "What is certain, is that M. Zermelo gives no method of *effectively* carrying out the operation of which he speaks [i.e. of making the infinitely many choices] and it remains doubtful whether anybody could finally indicate such a method. But the question of *effectively* giving a function is different from that of proving its existence—there is all the fundamental difference between them which there is between a *correspondence* which can be *defined* and one which can be described. Many important mathematical questions would completely change their sense if one word were to be substituted for the other." He also makes the point that the notion of a correspondence which can be described is not capable of precision, and belongs to psychology rather than to mathematics.

In the second letter Baire, writing to Hadamard, does not accept the latter's contention that in Zermelo's proof the successive choices are after all independent of one another for this is only accomplished by supposing that every subgroup of the set which is being well ordered has been made to correspond to one of its elements. He suggests that the set of those chosen elements cannot be regarded as 'given'. "In speaking of the infinite (even when enumerable . . .) the conscious or unconscious identification of the set with a bag of notes which can be given from hand to hand must disappear completely and, in my opinion, we are in the domain of the virtual, i.e. we make conventions which permit us eventually to make assertions about an object when this object has been defined *by a new convention*." Thus Baire explicitly states that it is false to consider the subsets of a given set as given.

Lebesque, in the third letter, is of the opinion that the existence of an entity is only proved when it has been defined, i.e. when a property characteristic of that entity has been given. He suggests that in vaguer cases of the use of the word existence, as by Zermelo, all that is meant is freedom from contradiction of the notions used. In general he supports Borel and Baire.

Hadamard, in the fourth letter, crystallizes the whole argument into the question "Can the existence of a mathematical entity be proved without defining it? I reply in the affirmative." He shows that the consequence of the opposing theories is the rejection of the whole Cantorian edifice of transfinite Alephs.

Borel, in the letter which closes this correspondence, accepts these drastic consequences and states that the only value of calculations employing the Aleph numbers is that they can provide suggestions for "more serious" demonstrations. Theorems in the Cantor theory of cardinals may, by analogy, be useful aids to the construction of valid proofs, but, in themselves, are statements with no precise meaning. They may, at most, have the status of certain theories in mathematical physics.

Intuitionism

There are two critically important points in Brouwer's doctrines concerning the nature of mathematics: the reduction of pure mathematics to an ultimate 'basal intuition' and the notorious 'denial'[1] of the *tertium non datur*. These aspects of the intuitionist philosophy are undoubtedly most difficult for those unfamiliar with this type of thought to understand and, if once sympathetically comprehended, remarkably facilitate the understanding of all that remains. The only contribution that the present writer can offer towards lightening the difficult effort of intellectual sympathy required—an effort materially increased by the imprecision of the terminology used by intuitionist expositors—is to point out with regard to the 'basal intuition' that Brouwer's views derive from and are a modification of Kant's (with alteration of terminology) and with regard to the remaining point that Brouwer denies only a reinterpretation of the logical principle in question. In fact Brouwer is a neo-Kantian who has rejected Kant's doctrines concerning space, while preserving his view of time as a pure intuition given a priori, and Brouwer's denial of the law of excluded middle is better interpreted as an emphasis, which can be paralleled in Kant, on the necessity for the constructibility of mathematical concepts.

The account which follows is divided into three sections:

(1) a sketch of the relations between Brouwer and Kant in so far as they bear on the 'basal intuition';

[1] ". . . Brouwer, the leader of what is called the intuitionist school, whose chief doctrine is the denial of the Law of Excluded Middle, that every proposition is either true or false." F. P. Ramsey: *Foundations of Mathematics*, p. 65.

(2) a description of Brouwer's sociological approach to science and the doctrine of the constructibility of mathematical concepts towards which it leads;

(3) the elaboration and technical consequences of Brouwer's doctrines.

Kant and Brouwer

Some striking analogies assist in understanding Brouwer.

The intuitionists bear much the same relation to the logisticians and formalists who preceded them as Kant's critical philosophy to the dogmatism which he attacked. Kant was concerned to rehabilitate philosophy after the destructive scepticism of Hume; the intuitionists, by setting out to explain in detail the anatomy of mathematics and the principles on which the understanding correctly functions, attempt to save mathematics from the destructive force of the mathematical paradoxes.

All the intuitionists agree in this, that they consider mathematical knowledge to be characterized by the employment of a specific method for obtaining knowledge, but differ among themselves as to the nature of the principle employed.

If, however, the views of Brouwer are considered, who is at once the most influential and the most consistent member of this school, it will be found that he bases mathematics on a primitive intuition, "a basal intuition of the bare two-oneness." What this means I shall try to make clear by examining the corresponding terms as they occur in Kant.

First, a few explanations of Kant's terminology. As is well known, he makes a distinction between *intellectual, empirical*, and *pure intuition*. By *empirical intuition* he means "*the immediate apprehension of a content which as given is due to the action of an independently real object upon the mind*" (N. K. Smith, *Commentary to Kant's Critique of*

Pure Reason, p. 80), and he terms all cognitive states *pure* in which there is nothing belonging to sensation. The following passage throws some light on the subject. "The pure form of *sensible intuitions* [apparently using *sensible intuitions* as synonymous with *empirical intuition* here] in general, in which all the manifold of intuition is intuited in certain relations, must be found in the mind a priori. This pure form of sensibility may also itself be called *pure intuition*. If then I take away from the representation of a body that which the understanding thinks in regard to it, substance, force, divisibility, etc., and likewise what belongs to sensation, impenetrability, hardness, colour, etc., something still remains over from this empirical intuition, namely extension and figure. These belong to pure intuition, which, even without any action of the senses or of sensation, exists in the mind a priori as a mere form of sensibility." (*Critique of Pure Reason*, p. 66). We need not linger over the somewhat misleading terminology here involved; it is important for our purpose to recall that for Kant, space and time are pure intuitions and therefore given a priori.

For him "space is not a discursive or, as we say, general concept of relations of things in general, but a pure intuition. For, in the first place, we can represent to ourselves only one space; and, if we speak of diverse spaces, we mean thereby only parts of one and the same unique space" (ibid., p. 69). This doctrine of space and the corresponding conception of geometry "as a science which determines the proportions of space synthetically and yet a priori" has now become obsolete by the discovery of non-Euclidean geometries, but the doctrine that time is a pure intuition is preserved by Brouwer. Thus he says "However weak the position of intuitionism seemed to be after this period of mathematical development it has recovered by abandoning Kant's apriority of space, but adhering the more resolutely to the apriority of time. This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers inasmuch as one of the elements of the two-oneness may be thought as a new two-oneness, which process may be repeated indefinitely ("Intuitionism and Formalism," Inaugural address at the University of Amsterdam, 1912).

The primary intuition of intervals of time as falling apart into sub-intervals, which may be resynthesized together to form the whole interval, is the basis of Brouwer's theory of the natural number. Brouwer's "Urintuition", the primitive intuition, approximates more to Kant's 'schema'. "If five points be set alongside one another thus . . . . ., I have an image of the number five. But if, on the other hand, *this thought is rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself*, this representation of a universal procedure of imagination in providing an image for a concept, I entitle the schema of this concept," [ibid., p. 182, my italics]. Thus the fundamental phenomenon on which Brouwer bases pure mathematics resembles what Kant called a *schema*, and their difference of nomenclature cannot obscure the profound similarities in their position. For Brouwer, as for Kant, the judgment of mathematicians are synthetic and a priori.

But Brouwer's improvements on the doctrine of Kant are seen in the former's insistence on the constructibility of mathematical entities. Kant sees the essence of philosophical knowledge in that its concepts are constructible. He says: "*Philosophical* knowledge is the knowledge gained by reason from the *construction* of concepts. To *construct* a concept means to exhibit a priori the intuition, which corresponds to the concept." (*Critique of Pure Reason*, p. 577), and again: "I construct a triangle by representing the object which corresponds to this concept either by imagination alone, in pure intuition—or in accordance therewith also on paper, in empirical intuition—in both cases completely a priori without having borrowed the pattern from any experience" (ibid.), and again: "mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents a priori, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object of the concept then constructed" (ibid., p. 578). This is altogether too vague to be regarded as a satisfactory account of the role of intuition in mathematics, but if we recall the part assigned to intuition in the formalist scheme we shall see how closely that view of mathematics also is related to the Kantian view of mathematical knowledge. For the formalist, too, the mathematical method is distinguished by the use of intuition but with this difference that the intuition can only function when the concepts have been embodied in concrete symbols. Thus the content of the formalist's intuitions is the relations between symbols, while the content of the mathematician's intuition, on Kant's view, consists of relations between concepts obtained from empiric intuitions of sense-data embodying those concepts. For Kant, then, geometrical results are to be obtained by intuitions derived from looking at triangles, circles, etc., drawn on paper. This view is obviously inadequate for modern geometry when such figures are quite unnecessary, and often physically impossible to represent.

Brouwer, however, arithmetizes the entire process and confines his 'basal intuition' to the form of the conceived multiplicity of the intervals of time. This process, according to him, is sufficient to generate the natural numbers, a series from which all other mathematical entities must be derived by modifications and repeated application of the same method.

The Sociological Basis of Mathematics

Brouwer's theory of the evolution of pure mathematics regards the laws of logic as the historical product of man's attempt to organize sets of object finite in number. On examination the same laws are found to apply also, with one exception, to the infinite subject matter of pure mathematics. That exception is the law of the excluded middle.

Brouwer bases his criticisms of certain logical and mathematical methods on historical and sociological grounds. In the present writer's opinion, questions of origin are irrelevant to the correctness of such methods and, though they may furnish presumptive evidence of the existence of errors, must always be supplemented by arguments concerned with the content, and not the history, of the criticized theories. Since, however, Brouwer himself considers arguments from origin of importance, it is as well to present his doctrines in the framework he has chosen. His sociological views are interesting on their own account, and undoubtedly reveal the surrounding atmosphere of his opinions.[1]

In his inaugural address at the University of Amsterdam (1912) Brouwer said: "To understand the development of

[1] Cf. especially, L. E. J. Brouwer, "Intuitionism and Formalism" (Inaugural address at the University of Amsterdam, 1912), reprinted in *Bulletin of the American Mathematical Society*, vol. xx, 1913, and "Mathematik, Wissenschaft und Sprache," *Monatshefte für Mathematik und Physik*, vol. xxxi, 1929, p. 153.

the opposing theories in this field [i.e. in the foundations of mathematics] one must first gain a clear understanding of the concept 'science'; for it is as a part of science that mathematics originally took its place in human thought." Science he conceives to be the systematic cataloguing as laws of nature of causal sequences of phenomena, especially such as are important in social relations. Mathematics, in particular, is a branch of scientific thought concerned with the structure of phenomena. A mathematical attitude towards phenomena arises as an act of will of the individual produced by an urge towards self-preservation, and the choice of structures for consideration is therefore determined by the exigencies of the individual in his relation to society. The earliest kinds of structure which men are forced to recognize are the forms of organization of the groups of persons with whom they live, the structure of society and the family; speech then arises as a medium for social activity, for the transference of wishes from individual to individual. A specifically scientific attitude arises in two stages, as a causal outlook and as a temporal outlook. In the first, men choose to consider phenomena in the aspect of identical self-repetition, a useful point of view because steadily improving catalogues of causal sequences of phenomena enables desired phenomena to be produced, knowledge of causes giving control over effects. Man not only discovers order in nature in this fashion but creates it by isolating causal sequences of phenomena, i.e. by experiment and construction. By his own ordered activity, he supplements the natural phenomena and widens the applicability of his laws. This is notably the case with counting and measuring, which are the activities *par excellence* by which man introduces order into nature. Mathematics, however, requires further explanation, for the causal laws so far described are essentially approximations and are not proof against sufficient refinement of measuring tools, in contrast to the unchangeable exactness of mathematics.

The origin of this exactness is the fact that mathematics arises out of the *temporal* outlook in which visual perceptions are regarded as separating into two parts (in the relation of before and after). From this is obtained the intuition of the primitive 'two-oneness', the whole as capable of division into two parts, in turn capable of division into two parts, and so on. The judgments of mathematics are synthetic and a priori, i.e. judgments independent of experience and not capable of analytic demonstration. This explains their apodictic exactness.

All this is plausible without being startling; and one may agree that mathematical activity has its roots in sociological activity while disagreeing profoundly with the intuitionist.

"The question where mathematical exactness exists, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper" (ibid.). For the formalist the fact that mathematical theorems are expressed in a symbolism is essential to the understanding of mathematical method; this is not the case with Brouwer. Language for him is necessarily uncertain and inexact. He asserts, moreover, that the use of language preceded in point of time the development of the scientific and mathematical outlooks. It was a natural consequence that, although the growth of mathematics demanded the invention of a new language of symbols to allow individuals to communicate with one another the results obtained, this new language adopted forms of grammatical convention of the old language of everyday activity.

"The laws of logic developed at a time in man's evolution when he had a good language for dealing with finite groups of phenomena. The so-called logical principles, therefore, arose as expressions of the structural interrelationships of sentences in the language, and later were found to work when applied to the universe, e.g. the law of the excluded middle was originally an hypothesis and when used in grounding such sciences as palæontology, etc., the practical impossibility of finding examples to disagree with the laws, replaced the 'logical impossibility' of the earlier laws of language. The reliability of logical principles, in practice, rests upon the fact that a large part of the universe of experience exhibits far more order and harmony [Treue und Zufriedenheit] in its finite organization than mankind itself" ("Mathematik, Wissenschaft u. Sprache," *Monats. für Math. u. Phys.*, vol. xxxi, 1929, p. 159).

On this somewhat dubious history is founded a distrust of the laws of logic. Formalism, from this point of view, appears to lay exaggerated emphasis on language, culminating in mistaken attempts to eliminate contradictions, without critical consideration of the particular concepts to which the laws of logic are being applied.

The forms of language are, however, mutable with its subject-matter, and the laws of logic amenable to critical investigation when applied to mathematical objects. In Brouwer's own words the result is favourable for the laws of identity, contradiction, and the syllogism, but unfavourable for the law of the excluded middle!

This then is the novelty of the whole position, but it must be considered in conjunction with the contention that only such mathematical entities 'exist' as can be constructed by means of the basal intuition.

"From the present point of view of intuitionism therefore all mathematical sets of units which are entitled to that name can be developed out of basal intuition, and this can only be done by combining a finite number of times the two operations: 'to create a finite ordinal number, and to create the infinite ordinal number ω[1]'; here it is to be

[1] ω is the ordinal number of the series 1, 2, 3, . . .

understood that for the latter purpose any previously constructed set or any previously performed constructive operation may be taken as a unit ".[1]

So, to summarize this account, Brouwer grounds intuition on an account of the historical development of the sciences. Science in general is characterized by two ways of classifying phenomena; first, by arranging them into causal sequences, secondly, by dividing them into parts which are in the temporal relations of immediately before and after; mathematics, in particular, arises out of the second process. Abstracting from the specific nature of the phenomena in any one such process of division in time gives the general scheme of ordinal succession out of which arises the basal intuition of the natural numbers. Logic, on the other hand, developed historically as the expression of the relationships between propositions, referring only to groups of phenomena finite in number. Hence its laws must not be assumed to hold for the infinite subject-matter of mathematics without further examination. The result of this examination shows that all the logical laws are valid except the law of the excluded middle.

The Denial of the Law of the Excluded Middle

In 'denying' the law Brouwer is emphatically asserting existence of mathematical entities to be synonymous with the possibility of their construction.

If the law is stated in the form that a proposition is either true or false its truth appears so obvious that it is incomprehensible that anybody should disbelieve it. But the apparent

[1] Consequently, the intuitionist recognizes only the existence of denumerable sets, i.e. sets whose elements may be brought into one-one correspondence either with the elements of a finite ordinal number or with those of the infinite ordinal number ω. And in the construction of those sets neither ordinary language nor any symbolic language can have any other part than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same set." "Intuitionism and Formalism," p. 86. See also p. 209 below.

simplicity of enunciation conceals the difficulties implicit in the notion of 'truth', a notion which those who most confidently believe in the law often find hardest to explain. In mathematics the question of the truth of mathematical theorems coincides with the question of the existence of mathematical entities; if the conditions of validity of mathematical theorems were known the conditions for the 'existence' of mathematical entities would be known and vice versa. So the dispute between Brouwer and more orthodox philosophers with respect to the validity of the *tertium non datur* is seen to be one as to the nature of mathematical existence rather than as to the validity of the logical principle. This interpretation gives body to the dispute and removes the air of paradox which surrounds Brouwer's philosophy. Brouwer, indeed, is not denying the *tertium non datur* in the generally accepted interpretation of that logical principle, but rather emphasizing that existence in mathematics is synonymous with constructibility, and that the truth, and indeed significance, of mathematical theorems is conditional on the possibility of constructing the entities which occur in their formulation. In order to understand his position fully it is therefore necessary to elucidate the notion of constructibility; this can be done by giving an account of Brouwer's treatment of the continuum.

The Intuitionist Continuum

'Points' in the continuum are obtained by using free-choice sequences constructed by arbitrary choices of integers at each stage; significant statements concerning such infinite sequences must contain an implicit or explicit indication of the method for testing their truth in a finite number of steps. This is the correct interpretation of the rejection of the law of the excluded middle.

We commence by describing a well known method for defining the points of a mathematical continuum by means of 'nests of intervals'. A 'nest' is a sequence of intervals each lying inside the previous one and contracting indefinitely in length; and each such nest picks out a real number from the continuum. If our continuum is a line we can for example divide it into the intervals $\ldots (-n-1, -n)$, $(-n, -n+1), \ldots (-1, 0), (0, 1), (1, 2), \ldots (n, n+1)$, $\ldots$ and then divide each of these into half, these new intervals again into half, and so on.

This process corresponds to the actual process of approximation in measurement. If we use instruments which measure with an outside error say of ·5 cm. we will be able to locate the position of any desired point inside an interval 1 cm. in length. Using more accurate instruments, say with an outside error of ·25 cm., we can now locate any desired point inside an interval ·5 cm. in length. Providing the outside error of our instruments ultimately becomes smaller than every length however small as we make them more and more accurate, we shall be able to specify any point by specifying a nest of these measurement intervals. In order to make the abstract scheme correspond better to this process of measurement the intervals at each end-stage must be made to overlap. Thus for the first stage we take the intervals $\ldots (-1, 0), (-\frac{1}{2}, +\frac{1}{2}), (0, 1), \ldots (n-\frac{1}{2}, n+\frac{1}{2}), \ldots$ and can now be sure that each point lies *inside* an interval of length of 1. With similar modifications at each end-stage we shall have a geometrical schema of the continuum.

To get the corresponding arithmetical schema we need only consider a geometrical schema in which there are a finite number of intervals at each stage, since this will supply arithmetical names for all the points in a finite stretch of the line, and we can set up a one-one correspondence between this stretch and the whole line, thus obtaining the names for all the points of the line.

These preliminaries accomplished, a point can be specified in the following way: at each stage of the geometrical schema we number off the finite set of intervals which have been constructed, and now specify a real number by stating the number of an interval which contains it at the *first* stage, the number of a smaller interval which contains it at the *second* stage, and so on. So each number is given by an infinite sequence of integers. This, of course, exactly corresponds to the specification of a point by means of a non-terminating decimal in the usual decimal representation.

All questions dealing with the existence of points on a line or the existence of real numbers may therefore be reduced to the existence of infinite sequences of integers, such sequences being constructed by arbitrary choices of integers at the first place, the second place, the third place, and so on. These are the 'Wahlfolge', Brouwer's 'arbitrary-choice sequences',[1] and the continuum is the concept whose denotation includes all such sequences. The continuum can, however, in no sense be said to be a complete totality, for though it can be more and more completely specified as our knowledge increases this brings us no nearer to exhausting it; it is a 'medium of free becoming'.

With respect to an infinite sequence of integers, generated by a succession of arbitrary choices, the intuitionists maintain that only those propositions are significant which can be verified in a finite number of operations. Any proposition which, for its verification, would necessarily involve the successive scrutiny of *all* the digits of the infinite sequence of digits is senseless just because the sequence is never finished. This excludes all general statements about the totality of integers in the sequence.

The theory can be illustrated by considering the slightly different case of a sequence whose successive digits are given by some kind of law, e.g. (i) the sequence of the digits in the decimal expansion of π, or (ii) the sequence of the

[1] Cf. *infra*, p. 203.

prime numbers in order of magnitude. In these cases it will be possible to make *some* general propositions, e.g. in (ii) we can say that all the places after the second are filled by odd numbers, but this is a sensible proposition only because the method of constructing the sequence allows us to verify the proposition in question in a finite number of steps. So, in general, it will not be permissible to make general statements about infinite sequences unless means are given of verifying them (or disproving them) in a finite number of steps, and *a fortiori* it will be impossible to make general statements about the continuum.

This I take to be the correct interpretation of the basis of the intuitionists' denial of the law of the excluded middle when applied to infinite sequences. How that denial follows from their peculiar view of the nature of general mathematical propositions it is easy to see. A general (or existential) proposition about the integers composing an infinite sequence can only be said to be true when a construction has been found which shows how to verify it in a finite number of steps. The corollary with respect to falsity of such propositions is equally important; they can be said to be false only when the assumption of their truth leads to a contradiction. If the 'truth' and 'falsity' of general mathematical propositions is interpreted in this way there is no reason to suppose that these two alternatives exclude all others, e.g. it may be impossible to prove Fermat's theorem and yet the assumption of its truth may lead to no contradictions. If this were the case we should have an example of a proposition neither false nor true. From this point of view the intuitionist position is based on the possibility of the existence of mathematical theorems which can neither be proved nor disproved, and has lately been strengthened by the discovery of the incompleteness of the calculus of propositional functions.

It follows that for the intuitionist the truth of a proposition

p is not, in general, equivalent to the falsity of its contradictory. Take, for instance, a proposition like (A): "there is a prime number of the form $x^4 + 1$" whose contradictory is, (B): "there is no prime number of the form $x^4 + 1$". The intuitionists will say that neither A nor B have sense until constructions are known for testing them. In this particular case we may, e.g., form (A'): "there is a prime number of the form $x^4 + 1$ and less than 18" whose verification is the simple one of testing whether any of the prime numbers which are less than 18 are of the form $x^4 + 1$. Similarly we might be able to form a B'. It does not follow that A' and B' will be contradictories in the orthodox sense.

I have set the matter out in this way in order to show that the conflict between Brouwer and supporters of traditional logic is one rather as to the correct criteria for the stating of mathematical propositions rather than any differences as to the validity of the *tertium non datur*. But there *is* a real difference of opinion between the intuitionists and those who take an extensional view of propositions existing in their own right.

In brief then, Brouwer's criterion of constructibility amounts to the statement that all genuine general propositions in mathematics must contain some method for verifying them in a finite number of steps; and the rejection of all forms of words which do not satisfy this condition leads to apparent denial of the law of the excluded middle. Consistent acceptance of this attitude demands reformulation of orthodox logic and of much orthodox mathematics, and this has been to a great extent accomplished with amazing energy and ingenuity by Brouwer and his disciples.

Supplementary Note on the Intuitionist Calculus of Propositions

This section and the next are of mainly technical interest; they include the complicated intuitionist definition of sets (classes).

A. Heyting has recently produced a calculus of intuitionist logic ("Die formalen Regeln der intuitionistichen Logik," *Sitzungsberichte der preussischen Akademie der Wissenschaften*, Phys.-Math. Kl., 1930, pp. 42–71, 158–169) and this account of the details of the intuitionist constructions is based partly on his paper and partly on the published work of Brouwer and Weyl.

It may be said at the outset that from the intuitionist point of view a calculus is useful merely as a means for understanding the ideas expressed by it and hence no particular emphasis is laid upon the semantic or meta-systematic concepts, involved in the study of the system *qua* object of investigation, which are all-important in the corresponding formalist structures.

Heyting uses four primitive concepts in the propositional calculus, viz. "a implies b", "a and b", "a or b", "not a", none of which can be defined in terms of the others. The sign "not a" or $\sim a$[1] may be better rendered perhaps as "a is impossible", for the calculus we are describing is meant to apply only to mathematical propositions. The chief difference between this calculus and the Russellian is that the formula $a \vee \sim a$ is not true. On the other hand $\sim\sim(a \vee \sim a)$ is a true formula. This is the so-called theorem of the absurdity of the absurdity of the law of the excluded middle.[2]

[1] The sign actually used by Heyting has been replaced here by $\sim$ for typographical reasons.

[2] It has been shown by V. Glivenko (*Bull. I. de Belgique*, 1929) that if a can be proved in ordinary logic, $\sim\sim a$ is a correct formula in the intuitionist logic, and that if $\sim a$ can be proved by ordinary logic, $\sim a$ is true in the intuitionist.

Heyting shows incidentally that the eleven axioms of the propositional calculus which he uses are independent of one another and that the *tertium non datur,* i.e. the formula $\sim\sim a \supset a$, cannot be proved from his axioms.

The intuitionist calculus of propositional functions contains some novel features. In particular, three different signs of equality or identity are used: (i) $p \equiv q$, "p is the same object as q", (ii) the sign $=$, for equality of numbers, etc., (iii) the sign $=$, used for mathematical identity (as distinct from equality) and defined afresh for each kind of mathematical object.

The formula $p \equiv p$ does not hold for all signs p, but is used to characterize those signs which stand for axioms; e.g. the axiom 6.1 reads $1 \equiv 1$, translated '1 is an object' (*sic*).

The chief divergences of this calculus from that of the logisticians[1] are (i) the fact that neither of the signs (x) or (Ex) can be defined in terms of the other, (ii) the introduction of the sign $(^p_x)a$ which may be translated as "the expression obtained from a when x, wherever it occurs in a, is replaced by the sign p".[2] In terms of the sign (^p_x) and $\equiv$ it is possible to define the expression $g\,(\bar{x})$ which may be translated as "g does not contain x". This makes it possible to distinguish between functional and propositional variables without using different kinds of letters for the two as in *Principia Mathematica.*

The properties of natural numbers are deduced from Peano's axioms, the following concepts being taken as primitive: ϵ, ("is a"); $x'p =$ ("the x of p") N ("natural number") *seq* ("successor of").

[1] Cf., e.g., Hilbert-Ackermann, *Gründgesetze der Logik.*

[2] The use of this sign seems to have been first introduced by Von Neumann ("Zur Hilbertschen Beweistheorie": *Mathematische Zeitschrift*, vol. xxvi, 1927).

We now come to the intuitionist definition of *sets*.[1] The definition of these, as of all mathematical entities in the intuitionist scheme, is based upon the basal intuition which supplies the infinite sequence of the natural numbers. Thus it is fundamental for the intuitionist definition of sets that we should be given an unending sequence of signs, defined by specifying the first member of the sequence and a law for deducing any member from the one which immediately precedes it. It is convenient to use for this purpose the sequence ζ of integers : 1, 2, 3, . . . It is the members of this sequence that Brouwer calls 'Nummern' in his definition of sets, and it is to them that reference will be made when speaking of 'integers' in what follows.

We have to consider infinite sequences of integers chosen at random, with repetitions permitted. In such sequences the members will in general obey no regular law of formation, and the sequence can be considered as constructed by successive arbitrary choices of an integer, each such choice being completely independent of the previous choices. Such an infinite sequence will be called a *choice-sequence* (Brouwer: 'Wahlfolge').

A set is a *law* which correlates groups of signs in the

[1] The definition as given by Brouwer is very obscure and is therefore reproduced here to guard against possible misinterpretation : "Eine *Menge* ist ein *Gesetz*, auf Grund dessen, wenn immer wieder eine willkürliche Nummer gewählt wird, jede dieser Wahlen entweder eine bestimmte Zeichenreihe mit oder ohne Beendigung des Prozesses erzeugt, oder aber die Hemmung des Prozesses mitsamt der definitiven Vernichtung seines Resultates herbeiführt, wobei für jedes $n > 1$, nach jeder unbeendigten und ungehemmten Folge von $n-1$ Wahlen, wenigstens eine Nummer angegeben werden kann, die, wenn sie als n-te Nummer gewählt wird, *nicht* die Hemmung des Prozesses herbeiführt. Jede in dieser Weise von einer unbegrenzten Wahlfolge erzeugte Folge von Zeichenreihen inklusive des charakters ihrer Fortsetzbarkeitsfreiheit, welche sich nach jeder Wahl beliebig (eventuell bis zur völligen Bestimmtheit, jedenfalls aber einem Mengengesetze entsprechend) verengern kann (welche also im allgemein nicht fertig darstellbar ist), heisst ein *Element der Menge*. Die gemeinsame Entstehungsart der Elemente einer Menge M werden wir kurz ebenfalls als *die Menge M* bezeichnen." (*Mathematische Annalen*, vol. lxxiii, p. 245.)

following fashion to *some* of all the possible arbitrary choice-sequences which can be obtained from the members of ζ: for any given specific arbitrary choice-sequence the law may (i) correlate some combination of signs to the first integer in the choice-sequence; this group of signs may be called the first stage in the element corresponding to that sequence. Or (ii) the law may specify that there is no group of signs correlated to the first integer.

If there *is* a first stage for the particular choice-sequence considered, the law may specify that the process ends at that stage, which is then the *final* stage for that sequence. If this is not the case we proceed to the second integer of the choice-sequence, for which the law may again correlate either (i) nothing, or (ii) a second stage which is final, or (iii) a non-final stage. If case (iii) arises we proceed to the third integer of the choice sequence, and so on. If, at any point of this procedure, case (i) arises then there is said to be no element corresponding to that particular sequence. Thus for any choice-sequence for which case (i) never arises at any stage we shall obtain a sequence of successive groups of signs correlated to the successive integers of the choice-sequence. If case (ii) arises at any stage the sequence of signs so obtained has a finite number of members; while if case (iii) always arises the sequence obtained has an infinite number of members. There is, however, one restriction on the above process which the law in question must conform to, viz. for each $n > 1$, if there is a choice-sequence, A say, for which there is a non-final $n - 1$ th stage then there is some choice-sequence B which has the same first $n - 1$ integers as A and has some nth (final or non-final) stage correlated to it.

The sequences of signs which are constructed in the above fashion are called *elements of the set*. It is a direct consequence of the method of defining a set that we can never completely specify all the elements of the set and may not be able to say, in general, whether some particular sequence of signs is an element of the set or not. Sometimes it is convenient to abstract from the particular choice-sequences to which the elements of a set correspond and to think of the set as the process which generates its elements. Thus in the first use of the term *set* we have in view the law by which the elements of the set can be constructed from choice-sequences; in the second use we emphasize rather what it is that these elements have in common, i.e. the manner in which all these elements can be obtained irrespective of the particular choice-sequences to which they are correlated.

Examples of such sets are (i) the set A whose elements are the integers of ζ. This set can be generated by the following simple law: "Every choice-sequence has a first stage which is final and is for each such sequence the integer which comes first in that sequence." (ii) C is the set of infinite sequences of integers, repetitions allowed. This could be generated by the following law. "For each choice-sequence the nth stage in the corresponding element is the nth integer of that choice-sequence, no stages being final."

In order to complete this account of what the intuitionists mean by set we must make precise what is meant by two elements of the same or different sets being identical and what is meant by two sets being identical.

Two elements of sets are said to be identical when we know that, for every n, the nth stages of both are the same combination of signs. Two sets are said to be identical when for each element of the one set an identical element of the other set can be given. Sets and elements of sets are called mathematical entities.

In addition to *sets*, the intuitionists have a hierarchy of *species* (Brouwer: *spezies*). The word *species* is roughly synonymous with property and is used in the following contexts: If x is a set $\mu'x$, the *set species of* x is the property which all those elements of sets possess which are identical with members of x. Thus being a set-species is the property which all x's possess which are members of some set y. Being a *species of order zero* is the property of being either an element of a set or a set-species. Being a *species of order one* is a property of all those properties which (*a*) can only be predicated of species of order zero and (*b*) if they hold for a species of order zero hold for all species of order zero which are identical with it.[1] Similarly *species of order x* may be defined. Species take the place in intuitionist mathematics of classes in the formalist and logistic developments of the subject.

The peculiar feature of the above definition of sets is our inevitable partial ignorance as to which signs are elements of the set. We shall know that some signs are definitely elements of the set and that other signs are definitely not elements of the set but there may be intermediate cases for which it is impossible to decide. This leads to much complexity. In the case of the mutual relationship of two sets, for instance, whereas in the classical theory of sets four cases arise according as whether the two sets do or do not, partially or wholly, include members of each other, the corresponding cases in intuitionist mathematics may be many more in number. It may be noticed first that two elements of a set are called *different* (Brouwen: *verschieden*) when it is impossible for them to be identical, i.e. when we are certain that it will never be possible, in the course of their development as sequences, to prove their equality. So for two sets M, N the following important cases may arise. (1) It may be impossible for M, N to be identical—we say M, N are *different*; (2) M *projects* (*herausragt*) out of N when N has an element which is different from all elements of M; (3) M, N are *congruent*

[1] The above definition of species differs from the one given by Brouwer in *Mathematische Annalen*, vol. xciii, p. 245, but is the simplified account given by Heyting (loc. cit., p. 167).

when neither can project out of the other, i.e. when every property which cannot possibly apply to the elements of the one cannot possibly apply to the elements of the other; (4) M, N may be said to be *exterior to one another* (*elementenfremd*) if they are different and it is impossible for an element of M to be identical with an element of N. There are exactly corresponding relations for species.

This complexity is even more apparent at later stages. Thus, owing to the complexity of the possible relationships between sets, when the intuitionist comes to define cardinal numbers as the common property of sets or species which can be put into one-one correspondence with one another, instead of the simple group of relationships $>$, $<$, $=$ which may hold between cardinal numbers as usually defined in mathematical textbooks, *four* such groups of relations appear, i.e. four different kinds of equality, etc. Fortunately these four groups have many of their properties in common.

Intuitionist Theory of Cardinal Numbers

In order to show clearly the divergences between the intuitionist view as to the validity of mathematical theorems and the more conventional ones it will be convenient to start with the classical theory of sets of points, for it is here that the Cantor theory of transfinite ordinals arises. It will incidentally become clear how the contradictions disappear for the intuitionist.

First to deal with Burali Forti's paradox of the greatest ordinal number. Some definitions are necessary. In what follows, we shall be meaning by 'set' what is meant by this word in the ordinary mathematical use of it, and *not* the special intuitionist meaning.

A set is said to be *ordered* if there is a serial relation R, such that it, or its converse, holds between every two elements of the set, R being such that aRb and bRc implies aRc. A *well-ordered* set is one for which every subset has a first member, i.e. which is a relatum with respect to R for no member of that subset. Two well-ordered sets, which can be brought into one-one correspondence in such a way that, if R is the ordering relation of the one set and R' of the other, and a, b any two elements of the one set and a', b' the corresponding elements of the other set, then $a'R'b'$ is true when and only when aRb is true, are said to have the same ordinal number. This account will be accepted in substance by the formalist, who will obtain ω, the ordinal number of the (well-ordered) series ζ (see p. 203), as his first infinite ordinal number.

To continue with the usual account of the matter, if two ordinal numbers A and B are not equal one is greater than the other, say B is greater than A. This means that A can be brought into a one-one correspondence (satisfying the conditions of the last paragraph) with a well-ordered subset of B. It follows quite simply from the above definitions that every subset of a well-ordered set is a well-ordered set whose ordinal number is less than, or equal to, that of the original set; also that if a new element is added to a well-ordered set in such a way that it is 'after' all the elements of the original set, the new set is well-ordered and has an ordinal number greater than that of the previous set. The last construction of course can easily be made precise.

Now on the classical view of the theory of sets, a set is well-defined if for every mathematical object it is determined whether it belongs to the set or not, from which we get the axiom that "*if for any mathematical object it is determined whether a certain property applies to it or not, then there exists a set containing nothing but those objects for which the property does hold*". This may be called the *axiom of inclusion.*

Burali-Forti's paradox now arises in the following way: consider the set S composed of all the ordinal numbers arranged in order of magnitude. This set, by the above *axiom of inclusion*, exists and can easily be shown to be well-ordered (by the ordering relation *greater than*). Hence it has an ordinal number which cannot be exceeded in magnitude by any other ordinal number. On the other hand, since not all mathematical objects are ordinal numbers, we may choose one, a say, which is not an ordinal number, and construct a well-ordered set S' by putting a at the end of S. S' will have a subset S and therefore an ordinal number greater than that of S. So the mathematician is faced with a blank contradiction.

This contradiction, however, cannot arise for the intuitionist who does not recognize the validity of the axiom of inclusion but builds up all his sets on the plan we have already described. The formalist, too, is forced to modify this axiom in order to avoid the paradox of the greatest ordinal. Thus Zermelo [1] replaces it by "If for all elements of a set it is determined whether a certain property is valid for them or not, then the set contains a subset containing nothing but those elements for which the property does hold" but can give no justification for so modifying it except that doing so will avoid this contradiction.

On the other hand, if the intuitionist is correct, nearly the whole of Cantor's theory of ordinal numbers is invalid. For example, on the classical view sets which have the same cardinal number as a set whose ordinal number is ω is called enumerably infinite and its cardinal number is called $\aleph_0$. Consider all those ordinal numbers of sets whose cardinal number is $\aleph_0$, and let us call these ordinal numbers the "denumerably infinite ordinal numbers". This is a concept which the intuitionist will allow as being clear and well defined. But, in the usual theory, it is shown that (*a*) sets with cardinal number $\aleph_0$ can be ordered in different ways to have various ordinal

[1] *Mathematische Annalen*, vol. lxv, p. 263.

numbers and (*b*) that for every denumerably infinite set of such ordinal numbers it is possible to assign a new "denumerably infinite" ordinal number not belonging to the set, from whence it is concluded that the set of denumerably infinite ordinal numbers has a cardinal number greater than $\aleph_0$. This cardinal is called $\aleph_1$ and the process is continued to obtain a whole series of cardinal numbers $\aleph_2$, $\aleph_3$, . . . $\aleph_\omega$, . . . corresponding to different sets of ordinal numbers. The intuitionist, however, while accepting propositions (*a*) and (*b*) says that the proposition "$\aleph_1$ is greater than $\aleph_0$" is without meaning. For the intuitionist indeed there is no infinite cardinal number except $\aleph_0$.

It follows from the above that the famous problem of the continuum, viz. the question whether *c*, the cardinal number of the number of points in a line or the number of real numbers in an interval, coincides with $\aleph_1$ or with one of the other of the $\aleph$ cardinals mentioned above, has no sense for the intuitionist.

Thus, if intuitionism is a correct theory, radical alterations are needed in pure mathematics, but it is unlikely that such a revolution will be accepted by practising experts until some agreement has been reached between logisticians, formalists, and intuitionists. And of such concord there is at present little sign. Our investigation may therefore suitably close with a question mark.

BIBLIOGRAPHY

Ackermann, W. "Begründung des 'tertium non datur' mittels der Hilbertschen Theorie der Widerspruchsfreiheit": *Mathematische Annalen*, vol. xciii, pp. 1–36. 1924.

—— "Ueber die Erfüllbarkeit gewisser Zahlausdrücke": *Mathematische Annalen*, vol. c, p. 638. 1928.

Baire, R., Borel, E., and others. "Cinq lettres sur la théorie des ensembles": *Bulletin de la Societé Mathématique de France*. 1905.

Bernays, P. "Die Philosophie der Mathematik und die Hilbertsche Beweistheorie": *Blätter für Deutsche Philosophie*, vol. iv, section 3–4.

—— "Axiomatische Untersuchung des Aussagen-Kalküls der *Principia Mathematica*": *Mathematische Zeitschrift*, vol. xxv, pp. 305–320. 1926.

Bernays, P., and Schönfinkel, M. "Zum Entscheidungsproblem": *Mathematische Annalen*, vol. xcix. 1928.

Bernstein, B. A. "*Principia Mathematica* (*review*)": Bulletin of American Mathematical Society, vol. xxviii, pp. 226–231. 1926.

Bolzano, B. *Wissenschaftslehre*. 1837; new edition, Leipsic, 1914.

—— *Paradoxien des Unendlichen*. 1851; new edition, Leipsic, 1920.

Boole, G. *The Mathematical Analysis of Logic*. Cambridge, 1847.

—— "The Calculus of Logic": *Cambridge Mathematical Journal*. 1854.

—— "*An Investigation of the Laws of Thought*" (reprinted London, 1916, in *Boole's Collected Works*). Dublin, 1854.

Brouwer, L. E. J. "Intuitionism and Formalism" (Inaugural address at the University of Amsterdam, 1912). Reprinted in *Bulletin of American Mathematical Society*, vol. xx. 1913.

—— "Zur Begründung der intuitionistichen Mathematik": *Mathematische Annalen*, vol. xciii, p. 244. 1925.

—— "Mathematik, Wissenschaft und Sprache": *Monatshefte für Mathematik und Physik*, vol. xxxi, p. 153. 1929.

Carnap, R. *Der Logische Aufbau der Welt*. Berlin, 1928.

Cassirer, E. *Philosophie der Symbolischen Formen*, vol. iii. Berlin, 1923–31.

Chwistek, L. *Theory of Constructive Types*. Cracow, 1925.

—— "Une méthode métamathématique d'analyse": *Comptes-Rendus du Premier Congrès des Mathématiciens des Pays Slaves*. Warsaw, 1929.

—— "Neue Grundlagen der Logik und Mathematik": *Mathematische Zeitschrift*. 1929 and 1932.

Couturat, L. *L'infini mathématique*. Paris, 1896.

CREEDY, F. "Logic as Cross Classification and selection of Arbitrary Elements": *Monist.* April, 1931.
CURRY, H. B. "An analysis of Logical Substitutions": *American Journal of Mathematics*, pp. 363–384. 1929.
DEDEKIND, R. *Stetigkeit und irrationale Zahlen.* Fifth Edition, Brunswick, 1918. Originally appeared in 1872.
—— *Was sind und was sollen die Zahlen?* Fourth Edition, Brunswick, 1918. Originally appeared in 1887.
EATON, R. M. *Symbolism and Truth.* Harvard, 1925.
FRAENKEL, A. *Einleitung in die Mengenlehre.* Third Edition, Berlin, 1928. Contains a valuable bibliography.
FREGE, G. *Die Grundlagen der Arithmetik.* Breslau, 1884.
—— *Grundgesetze der Arithmetik*, Jena: vol. i, 1893; vol. ii, 1903.
GÖDEL, K. "Die Vollständigkeit der Axiome der *Principia Mathematica*, etc.": *Monatshefte für Mathematik und Physik.* 1930.
—— "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwand. Systeme, I": Ibid. 1931.
—— "Die Vollständigkeit der Axiome des logischen Funktionenkalkül": Ibid., vol. xxxvii, pp. 349–360. 1930.
HEYTING, A. "Die formalen Regeln der intuistischen Logik": *Sitzungsberichte der preussischen Akademie der Wissenschaft.* Berlin, 1930. Phys.-Math. Klasse, pp. 42–56; 57–71; 158–169.
HILBERT, D. *Grundlagen der Geometrie.* Seventh Edition, Berlin, 1930.
—— "Ueber die Grundlagen der Logik und Arithmetik": *Verhandlungen d. 3 intern. Math. Kongresses in Heidelberg.* 1904.
—— "Axiomatisches Denken": *Mathematische Annalen*, vol. lxxviii. 1918.
—— "Ueber das Unendliche": Ibid., vol. xcv. 1925.
—— "Die Grundlagen der Mathematik": *Abh. aus d. Math. Sem. d. Hamb. Univ.*, vol. vi. 1928.
HILBERT, D., and ACKERMANN, W. *Grundzuge der theoretischen Logik.* Berlin, 1928.
JOHNSON, W. E. *Logic.* Three Parts, Cambridge, 1921–1924.
JØRGENSEN, J. *A Treatise of Formal Logic.* Three volumes, Oxford, 1931.
KAUFMANN, F. *Das Unendliche in der Mathematik und seine Ausschaltung.* Vienna, 1930.
KRONECKER. "Ueber den Zahlbegriff": *Journ. für Reine u. Angewandte Math.*, vol. ci, p. 337. 1887.
LANDAU, E. *Grundlagen der Analysis.* 1929.
LESNIEWSKI, ST. "Grundzüge eines neuen Systems der Grundlagen der Mathematik": *Fundamenta Mathematica*, pp. 1–81. 1929.
LEWIS, C. I. *A Survey of Symbolic Logic.* Berkeley, 1918.
LEWIS, C. I., and LANGFORD, C. H. *Symbolic Logic.* New York, 1932.
LITTLEWOOD, J. E. *The Elements of the Theory of Real Functions.* Cambridge, 1926.

LÖWENHEIM, L. "Ueber Moglichkeiten im Relativkalkül": *Mathematische Annalen*, vol. lxxvi. 1915.
LUKASIEWICZ, J., and TARSKI, A. "Untersuchungen über den Aussagenkalkül": *Comptes-Rendus des Séances de la Societé des Sciences et des Lettres de Varsovie*, vol. xxiii, class iii, pp. 1–21. 1930.
LUKASIEWICZ, J. "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls": Ibid., pp. 51–77.
MENGER, K. "Der Intuitionismus": *Blätter für Deutsche Philosophie*, vol. iv, Heft 3–4.
—— "Bemerkungen zu Grundlagenfragen": *Jahresberichte d. Deutschen Math. Ver.*, vol. xxxvii. 1928.
MOORE, G. E. *Philosophical Studies*. London, 1922.
DE MORGAN, A. *Formal Logic: or the Calculus of Inference, Necessary and Probable*. 1847; Edited by A. E. Taylor, 1926.
VON NEUMANN, J. "Zur Hilbertschen Beweistheorie": *Mathematische Zeitschrift*, vol. xxvi. 1927.
NICOD, J. "A reduction in the number of the Primitive Propositions of Logic": *Proc. Cam. Phil. Soc.*, vol. xix, pp. 32–41.
PEANO, G. *Formulaire de Mathématiques* (in collaboration). Five editions from 1893–1908. Turin.
PEIRCE, C. S. *Collected Papers*. Edited by Charles Hartshorne and Paul Weiss. Cambridge, Mass. 1931–.
PLANCK, M. *The Origin and Development of the Quantum Theory*. Translated by H. T. Clarke and L. Silberstein. Oxford, 1922.
POINCARÉ, H. *Science and Hypothesis*. Translated by W. J. G. London and Newcastle-upon-Tyne, 1905.
—— *Science and Method*. Translated by Francis Maitland. London, 1914.
—— *La Valeur de la Science*. Paris, 1904.
RAMSEY, F. P. *The Foundations of Mathematics*. Kegan Paul, 1931.
RUSSELL, B. A. W. *The Principles of Mathematics I*. Cambridge, 1903.
—— "Mathematical Logic as based on the Theory of Types": *American Journal of Mathematics*, vol. xxx. 1908.
—— *Introduction to Mathematical Philosophy*. London, 1919.
—— *Our Knowledge of the External World*. Revised edition, London, 1926.
RUSSELL, B. A. W., and WHITEHEAD, A. N. *Principia Mathematica*. Cambridge. Three volumes. Vol. i, 1910; revised edition, 1925. Vol. ii, unaltered second edition, 1927. Vol. iii, 1913; unaltered second edition, 1927.
SCHRÖDER, E. *Vorlesungen über die Algebra der Logik*. Three volumes, Leipzig, 1890–1905.
SHEFFER, H. M. "*Principia Mathematica*": *Isis; Quarterly Organ of the History of Science Society*, vol. viii (i), pp. 226–231. 1926.
—— Review of C. I. Lewis's *A Survey of Symbolic Logic*: *American Mathematical Monthly*, vol. xxvii, pp. 309–311. 1920.

SKOLEM, TH. "Logische-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze": *Shrifter utgit av Videnskaposelskapet i Kristiana*, i, Mat.-naturv. Klasse No. 4. 1920.

STEBBING, L. S. *A Modern Introduction to Logic.* London, 1930.

TARSKI, A. "Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften, I": *Monatshefte für Mathematik und Physik,* vol. xxxvii, pp. 349–360. 1930.

WAISMANN, F. "Die Natur des Reduzibilitätsaxioms": *Monatshefte für Mathematik und Physik,* vol. xxxv. 1928.

WEISS, P. "The Theory of Types": *Mind,* vol. xxxvii. 1928.

WEYL, H. *Das Kontinuum.* Leipzig, 1918.

—— "Der circulus vitiosus in der heutigen Begründung der Analysis": *Jahresb. d. Deutschen Mathematiker-Vereinigung,* vol. xxviii. 1919.

—— *Philosophie der Mathematik und Naturwissenschaften.* (Section IIA of the *Handbuch der Philosophie,* edited by A. Baeumler and M. Schröter). Munich and Berlin, 1926.

WHITEHEAD, A. N. *Universal Algebra.* Cambridge, 1898.

WISDOM, A. J. T. D. *Analysis and Interpretation.* Kegan Paul, London, 1931.

WITTGENSTEIN, L. *Tractatus Logico-Philosophicus.* London, Kegan Paul, 1922.

—— "Some Remarks on Logical Form": *Aristotelian Society Supplementary Volume IX.* 1929.

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